EECE 301 NETWORK ANALYSIS II Dr. Charles J. Kim

Class Note 01: **Laplace Transformation**

1. Integral Transformation:

Laplace transformation belongs to a class of analysis methods called integral transformation which are studied in the field of operational calculus. These methods include the Fourier transform, the Mellin transform, etc. In each method, the idea is to transform a difficult problem into an easy problem. For example, taking the Laplace transform of both sides of a linear, ordinary differential equation results in an algebraic problem. Solving algebraic equations is usually easier than solving differential equations. The one-sided Laplace transform defined in 3 below, is valid over the interval [0, ∞). This means that the domain of integration includes its left end point. This is why most authors use the term (0-) to represent the bottom limit of the Laplace integral.

2. Comparison of a few Transform methods:

A. FOURIER TRANSFORM: $X(f)=\int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$

(1) Usually for energy signals, in the limit for singularity functions, periodic signals, causal or non-causal signals.
(2) Steady state circuit analysis, algebraic differential solutions.
(3) It can be used to perform convolution very fast for discrete signals.
(4) Fourier transform is a function of one variable, w, and plots of it have a lot of meaning.

B. LAPLACE TRANSFORM: $X(s)=\int_{0-}^{\infty} x(t)e^{-st}dt$

(1) Usually for signals starting at time zero, exists for many non-energy, non-power signals for which the Fourier transform does not exists.
(2) Transient and steady state analysis; initial conditions in differential equations handled algebraic differential equations.
(3) Analysis tool, but not useful for fast convolution.
(4) Laplace transform is function of one complex variable, g, much harder to plot and the plots have much less usefulness.

C. Z-TRANSFORMATION : $Y(z)=\sum_{k=0}^{\infty} y(k)z^{-k}$

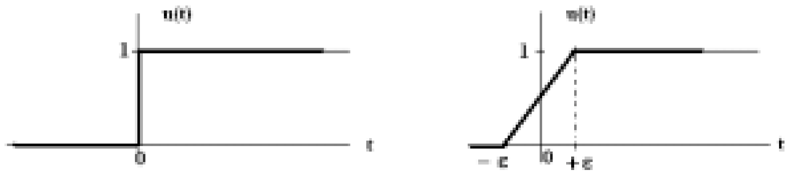
(1) The method of z-transformation does for discrete systems what Laplace transformation does for continuous systems.

3. Definition:

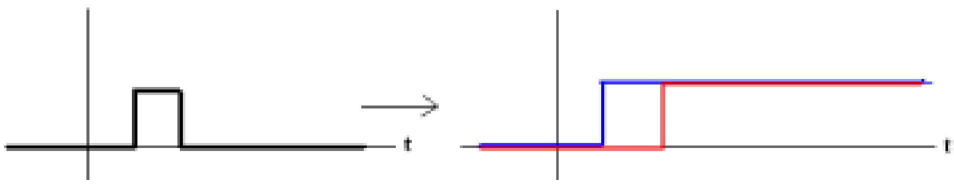
$$L(f(t)\}=\int_{0-}^{\infty} f(t)e^{-st}dt = F(s)$$

4. Signals of Interest

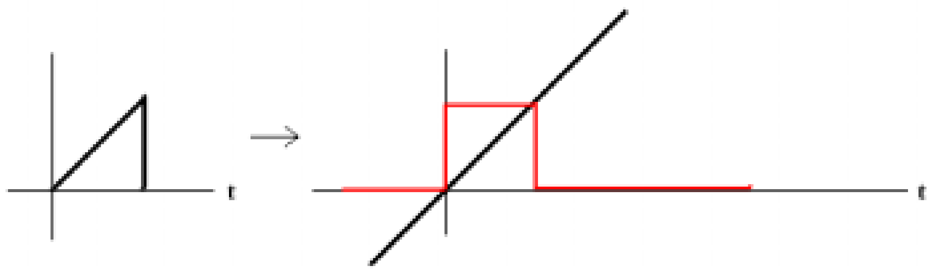
a. Step Function



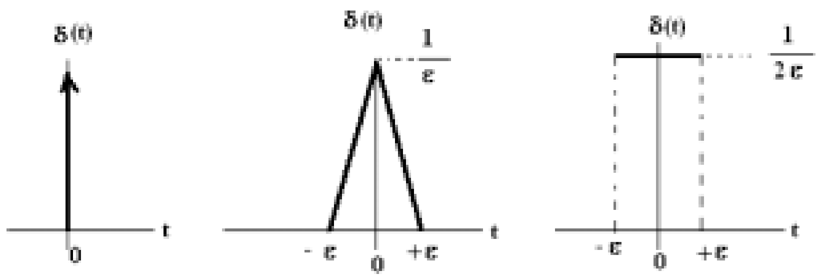
b. Rectangular Function expressed using Step function



c. Expression of a function using step function



d. Delta function



5. Laplace transformation of selected function:

a. $L\{\delta(t)\} = 1$

b. $L\{u(t)\} = \dfrac{1}{s}$

c. $L\{e^{-at}\} = \dfrac{1}{s+a}$

d. $L\{\cos wt\} = \dfrac{s}{s^2 + w^2}$

e. $L\{\sin wt\} = \dfrac{w}{s^2 + w^2}$

f. $L\{t\} = \frac{1}{s^2}$

g. $L\{t^2\} = \frac{2}{s^3}$ ------> *general form: $L\{t^n\} = \frac{n!}{s^{n+1}}$

h. $L\{\delta'(t)\} = s$

L'Hopital's Rule

i). **Guillaume Francois Antoine de L'Hopital (1661 – 1704)** was a French analyst and geometer. He wrote the first textbook on differential calculus.

ii). L'Hopital's Rule: A rule to evaluate indeterminate forms:

IF $\lim_{x \to a} f(x) = \lim_{x \to a} F(x) = 0$ OR $\lim_{x \to a} |f(x)| = \lim_{x \to a} |F(x)| = \pm\infty$

AND $\frac{f'(x)}{F'(x)}$ approaches a limit as x approaches a,

THEN $\frac{f(x)}{F(x)}$ approaches the same limit.

iii). L'Hopital's Rule can be proved under the assumption that there is a neighborhood U of a such that f and F are differentiable in U except possibly at a and there is no point of U at which f' and F' are both zero.

iv). Example: If $f(x) = x^2 - 1$, $F(x) = x - 1$, and a=1, then, $\frac{f(a)}{F(a)}$ takes the form of $\frac{0}{0}$.

Since $\lim_{x \to 1} \frac{f'(x)}{F'(x)} = \lim_{x \to 1} 2x = 2$, therefore, $\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$

v). They say that L'Hopital's Rule was actually discovered by John Bernoulli and given to L'Hopital in return for salary.

6. Properties of Laplace Transformations:

a. Linearity: $L\{kf(t)\} = kF(s)$

b. Scaling: $L\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$

c. Time Shift: $L\{f(t-a)u(t-a)\} = e^{-as} F(s)$

*also, $L\{u(t-a)\} = \frac{e^{-as}}{s}$

d. Frequency Shift: $L\{e^{-at} f(t)\} = F(s+a)$

e. Time Differentiation: $L\{f'(t)\} = sF(s) - f(0^-)$

*general form: $\frac{d^n f}{dt^n} = s^n F(s) - s^{n-1} f(0^-) - s^{n-2} f'(0^-) - \ldots - f^{(n)}(0^-)$

f. Time Integration: $L\{\int_0^t f(x)dx\} = \frac{F(s)}{s}$

g. Frequency Differentiation: $L\{t \cdot f(t)\} = -F'(s)$

h. Frequency Integration: $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)du$

i. Time Periodicity: $L\{f(t) = f(t+nT)\} = \frac{F_1(s)}{1-e^{-Ts}}$

j. Initial Value: $f(0^+) = \lim_{s\to\infty} sF(s)$

k. Final Value: $f(\infty) = \lim_{s\to 0} sF(s)$

l. Convolution**:** $L\{f(t) * g(t)\} = F(s)G(s)$