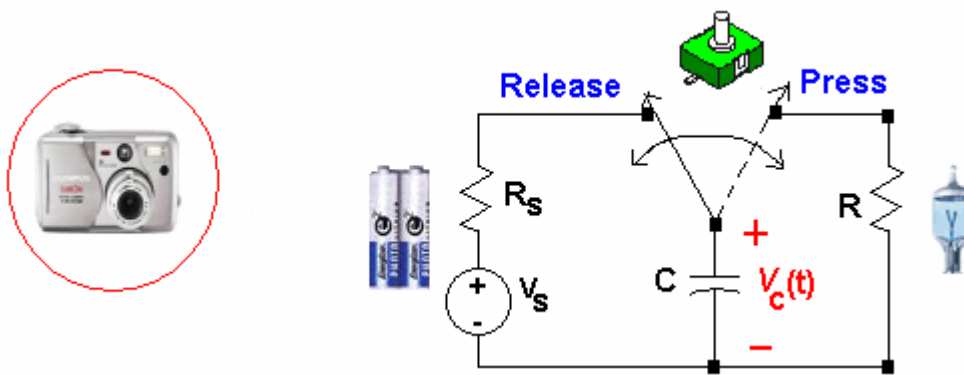


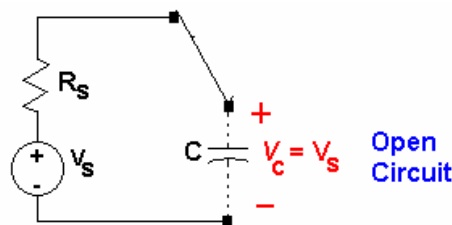
**Class Note 24: First Order Transient Circuit**

**A. Example First**

1. Before we solve a first order differential equation, let's consider an example circuit.
2. Consider the flash circuit in a camera. The operation of the flash circuit, from a user standpoint, involves depressing the push button on the camera that triggers both the shutter and the flash and then waiting a few seconds before repeating the process to take the next picture.
3. This operation can be modeled using the circuit below. The voltage source  $V_s$  and the resistor  $R_s$  model the battery that power the camera and flash. The capacitor models the energy storage, the switch models the push button. And the resistor  $R$  models the xenon flash lamp.

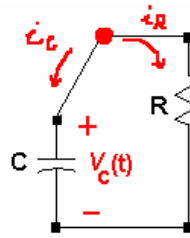


4. The capacitor is charged when push button is in the released position.
5. When the button is pressed, the capacitor energy is released through the xenon lamp, producing the flash. In practice, this energy release takes about 1 ms and the discharge time is a function of the elements in the circuit.
6. When the button is released, the battery recharges the capacitor. Again, the time required to charge the capacitor is a function of the circuit elements.
7. Let's further investigate the charging and discharging of the capacitor:
  - (a) Charging the capacitor (push button in released position): In DC circuit, the capacitor is an open circuit, therefore, no current flows through the circuit. Hence, the voltage across the capacitor is same as the source voltage.



(b) Discharging the capacitor (push button pressed): when the switch is closed, the node

voltage equation, from  $i_c + i_R = 0$ , is:  $C \frac{dv_c(t)}{dt} + \frac{v_c(t)}{R} = 0$  ---(A.1)



(c) The equation (A.1) is rearranged as:  $\frac{dv_c(t)}{dt} + \frac{1}{RC} v_c(t) = 0$  ----(A.2)

8. If we write the equation (2) using a general variable  $x(t)$ , instead of the voltage variable,  $v_c(t)$ , we could have the following a general first order differential equation, which is the subject of the next section.  $\frac{dx(t)}{dt} + ax(t) = 0$  -----(A.3)

## B. General solution of a first-order differential equation

1. Let's start from a first order differential equation of the form

$$\frac{dx(t)}{dt} + ax(t) = A, (A \text{ is some constant})\text{----(B.1)}$$

\*: Note that  $a$  and  $A$  in (B.1) correspond to  $\frac{1}{RC}$  and  $0$ , respectively, in (A.2)

2. A fundamental theorem of differential equation states that:

IF

$$x(t) = x_p(t) \text{ is any solution of equation } \frac{dx(t)}{dt} + ax(t) = A \text{ ---(B.1)}$$

AND IF

$$x(t) = x_c(t) \text{ is any solution to the homogeneous equation } \frac{dx(t)}{dt} + ax(t) = 0 \text{ ---(B.2)}$$

THEN

$$x(t) = x_p(t) + x_c(t) \text{ ----(B.3) is a solution to the original equation (B.1)}$$

3. The term  $x_p(t)$  is called the **particular integral solution** (or forced response), and the term  $x_c(t)$  is called the **complimentary solution** (or natural response).

4. Then, the general solution of the equation (B.1) consists of two parts that are obtained by solving the two equations:

$$\frac{dx_p(t)}{dt} + ax_p(t) = A \text{ ----(B.4)}$$

$$\frac{dx_c(t)}{dt} + ax_c(t) = 0 \text{ ----(B.5)}$$

5. Since the right-hand side of equation (B.4) is a constant, it is reasonable to assume that the solution  $x_p(t)$  must also be a constant. Therefore, we assume that:  $x_p(t) = K_1$  ----(B.6)

6. Substituting this constant in to equation (B.4) yields:  $K_1 = \frac{A}{a}$  -----(B.7)

7. Equation (B.5) can be rearranged to:  $\frac{dx_c(t)/dt}{x_c(t)} = -a$  -----(B.8)

8. The equation (B.8) is equivalent to:  $\frac{d}{dt}[\ln x_c(t)] = -a$  -----(B.9)

9. Therefore, equation (B.9) becomes:  $\ln x_c(t) = -at + c$  -----(B.10)

10. Equation (B.10), then, becomes:  $\ln x_c(t) = \ln[e^{-at+c}] = \ln[e^{-at} \cdot e^c] = \ln[K_2 \cdot e^{-at}]$

11. Therefore,  $x_c(t) = K_2 \cdot e^{-at}$  -----(B.11)

12. Finally, the solution to equation (B.1) is:

$$\underline{x(t) = K_1 + K_2 e^{-at}} \text{ -----(B.12)}$$

13. The constant  $K_2$  (also  $K_1$ ) can be found if the value of the independent variable  $x(t)$  is known at two instances of time.

Let's evaluate  $x(t)$  at  $t = t_0$ :  $x(t_0) = K_1 + K_2 e^{-at_0}$

Also at  $t = \infty$ :  $x(\infty) = K_1 + K_2 e^{-a\infty} = K_1 = \frac{A}{a}$  <----from (B.7)

From above two equations, we could get for  $K_2$ :  $K_2 = K[x(t_0) - x(\infty)]e^{at_0}$

14. Then, we can rewrite the solution (B.10) in to:

$$\underline{x(t) = K_1 + K_2 e^{-at} = x(\infty) + [x(t_0) - x(\infty)]e^{-a(t-t_0)}} \text{ (B.11)}$$

15. If we choose  $t_0=0$ , then the solution becomes:

$$\underline{x(t) = K_1 + K_2 e^{-at} = x(\infty) + [x(0) - x(\infty)]e^{-at}} \text{ (B.12)}$$

16. In plain term, we could say this way:

$$\text{(Solution)} = \text{(Final value)} + [(\text{initial value}) - (\text{final value})] \exp[-at]$$

### C. Camera Flash Circuit Case

1. Let's go back to the camera flash circuit to apply the general solution for the first order differential equation.

2. The node-voltage equation across the capacitor is given by:  $\frac{dv_c(t)}{dt} + \frac{1}{RC} v_c(t) = 0$

3. We see that:  $a = 1/RC$  and  $A = 0$  ----  $K_1 = A/a = 0$ .

4. Therefore, the voltage across the capacitor is in the form  $v_c(t) = [v_c(0) - v_c(\infty)]e^{-t/RC}$

5. From the circuit, the initial voltage  $v_c(0) = V_s$

6. Also, from the circuit, the final voltage at time  $t = \infty$ , the voltage will die eventually because there is no voltage source in the circuit but a consuming resistor. So  $v_c(\infty) = 0$ .

[We can calculate this value by using (B.7):  $v_c(\infty) = \frac{A}{a} = 0$ ]

6. Finally,  $v_c(t) = [V_s - 0]e^{-t/RC} = V_s e^{-t/\tau}$ , where time constant  $\tau = RC$

## D. Time Constant

1. Let's consider a differential equation of:  $\frac{dx(t)}{dt} + \frac{1}{\tau}x(t) = 0$ . Then the solution form is:

$$x(t) = x(\infty) + [x(t_0) - x(\infty)]e^{-(t-t_0)/\tau}$$

2. Assuming that  $t_0=0$  yields to:

$$x(t) = x(\infty) + [x(0) - x(\infty)]e^{-t/\tau}$$

4. From  $x(\infty)=0$ , it further reduces to:

$$x(t) = x(0) \cdot e^{-t/\tau}$$

3. The rate of the decay is determined by the constant  $\tau$ , “**time constant**”.

4. Let's find the values of the decaying term  $x(t)$  at time  $t=0$  and  $t=\tau$ :

$$x(0) = x(0) \cdot e^{-0/\tau} = x(0)$$

$$x(\tau) = x(0) \cdot e^{-\tau/\tau} = x(0) \cdot e^{-1}$$

5. By comparing the values of the decaying term, we have:  $\frac{x(\tau)}{x(0)} = \frac{x(0)e^{-1}}{x(0)} = e^{-1}$

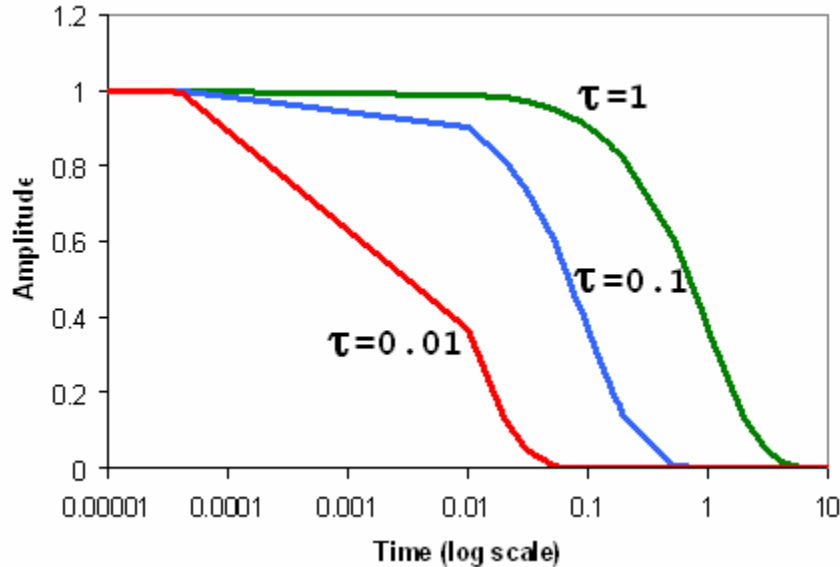
6. The time constant of a circuit then is defined as the time required for the response to decay by a factor of  $1/e$  (or 0.368 or 36.8%) of its initial value.

7. The value of  $\frac{x(t)}{x(0)} = \frac{x(0)e^{-t/\tau}}{x(0)}$  for several  $t$  values in terms of  $\tau$  is tabulated for a graph:

The value of $\frac{x(t)}{x(0)} = e^{-t/\tau}$	
time $t=$	$e^{-t/\tau}$
0	$e^0=1.00000$
$\tau$	$e^{-1}=0.36788$
$2\tau$	$e^{-2}=0.13534$
$3\tau$	$e^{-3}=0.04979$
$4\tau$	$e^{-4}=0.01832$
$5\tau$	$e^{-5}=0.00674$

8. A graph and observations: It is evident that the value  $x(t)$  is less than 1% of the initial value after  $5\tau$  (i.e., five time constants). Thus, it is customary to assume that it takes  $5\tau$  for the circuit to reach its final (or steady) state.

9. Another observation: The smaller the time constant, the more rapidly the value  $x(t)$  decreases.



### E. Transient Circuits (SUMMARY)

1. The state transition occurs when we suddenly apply to, or instantly remove from, a circuit the source of energy.
2. The analysis of the circuit behavior in the transition phase is called a transient analysis.
3. The transient behavior of circuits is affected by the presence of capacitor or inductor, or both, since these two elements can store energy and releasing it over some interval of time.
4. A circuit comprising a resistor and a capacitor (“RC circuit”), and a circuit comprising a resistor and an inductor (“RL circuit”), result in a first order differential equation.
5. A first order circuit is characterized by a first order differential equation.
6. There are two ways to excite a circuit.
  - (a) By initial conditions of the storage elements in the circuit: in this *source-free circuit*, we assume that energy is initially stored in the capacitive or inductive elements of the circuit and the energy causes current to flow in the circuit and is gradually dissipated in the resistors.
  - (b) By independent sources: Only DC sources are considered in the course.
7. The natural response of a circuit refers to the behavior (voltages and currents. What else?) of the circuit itself, with no external sources of excitation.
8. The step response of a circuit is its behavior when the excitation is the step function, which may be a voltage or a current source.
9. The two types of first-order circuits and the two ways of exciting them add up to the four possible situations:
  - (a) Natural response of RC circuit
  - (b) Natural response of RL circuit
  - (c) Step response of RC circuit
  - (d) Step response of RL circuit

## F. First-Order Circuit Examples-----Review

F.1. A general first-order differential equation:  $\frac{dx(t)}{dt} + \frac{1}{\tau}x(t) = A$ .

F.2. Then the solution form is:  $x(t) = x(\infty) + [x(t_0) - x(\infty)]e^{-(t-t_0)/\tau}$  with  $x(\infty) = \tau \cdot A$

F.3. We analyze four categories of first-order circuits. They are

- (a) RL Natural Response
- (b) RC Natural Response
- (c) RL Step Response
- (d) RC Step Response

## G. RL Natural Response Example

G.1. Summary

(a) Circuit formation: RL Parallel circuit with initial current in the inductor.  
No source after  $t > 0$ .

(b) Linear first order differential equation:  $L \frac{di}{dt} + Ri = 0 \rightarrow \frac{di}{dt} + \frac{R}{L}i = 0$ ,  $x(\infty) = \tau \cdot A = 0$

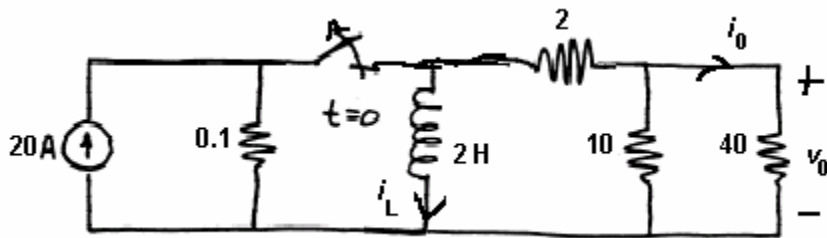
(c) Solution form:  $i(t) = i(t_0) \cdot e^{-\frac{t}{\tau}} = i(t_0)e^{-\frac{Rt}{L}}$ ,  $\tau = \frac{L}{R}$

(d) Power equation:  $p = vi$

(e) Energy equation:  $w = \int_0^t p dx$

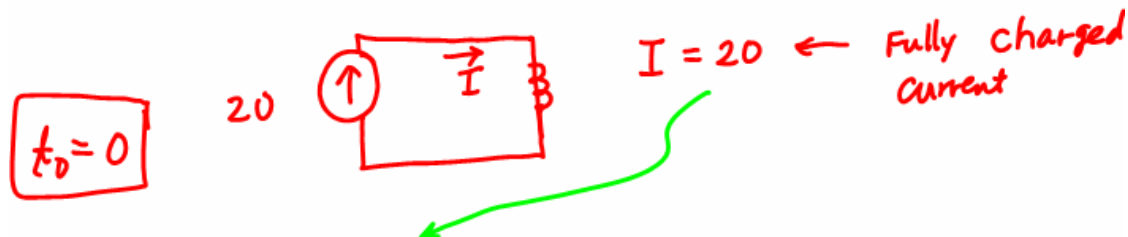
G.2. Example: The switch has been closed for a long time before being opened at  $t=0$ .

Find  $i_L(t)$ ,  $i_0(t)$ , and  $v_0(t)$ . Also find  $P_{10\Omega}$  and  $w_{10\Omega}$  for  $t > 0$ .

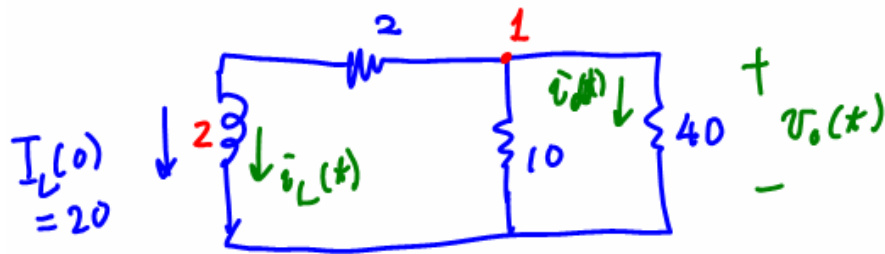


SOLUTION:

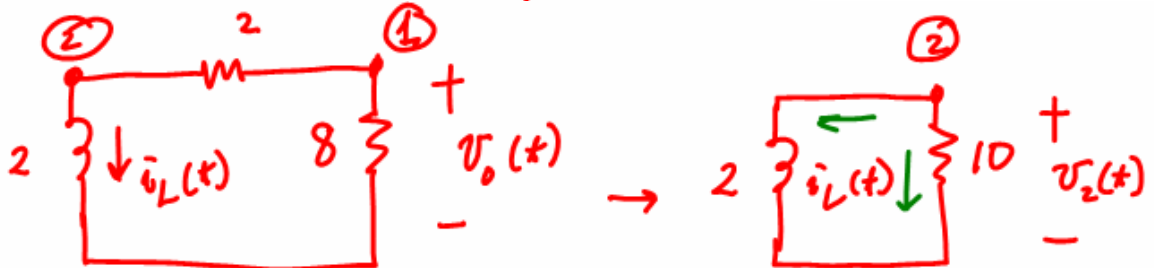
$t < 0$ :  $L$  is fully charged.  
(In steady-state,  $L$  is short circuit.)



$t > 0$ :  $I_L(0) = 20 \text{ [A]}$  initial change



↓ Simplify



$$i_L(t) + \frac{v_2(t)}{10} = 0 \quad \leftarrow \quad v_2(t) = 2 \frac{di_L(t)}{dt}$$

$$\tau = \frac{L}{R} = \frac{2}{10} = 0.2 \text{ s}$$

$$\frac{2}{10} \frac{di_L(t)}{dt} + i_L(t) = 0 \quad \rightarrow \quad \frac{di_L(t)}{dt} + 5i_L(t) = 0$$

From Sol form

$$i_L(t) = i_L(\infty) + [i_L(0) - i_L(\infty)] e^{-5t}$$

$A=0$   
or  $\rightarrow i_L(\infty) = 0$

$\therefore$  Energy stored will be all discharged

$$= \underline{20 e^{-5t}}$$

$$\frac{1}{\tau} = \frac{1}{0.2} = 5$$

$$v_o(t) = -8 \cdot i_L(t) = \underline{-160 e^{-5t}}$$

$$i_o(t) = \frac{v_o(t)}{40} = \underline{-4 e^{-5t}}$$

## H. RL Step Response Example

### H.1. Summary

(a) Circuit formation: R-L Series. No source before  $t=0$ .

(b) DC voltage source ( $V_s$ ) is suddenly applied at  $t=0$ .

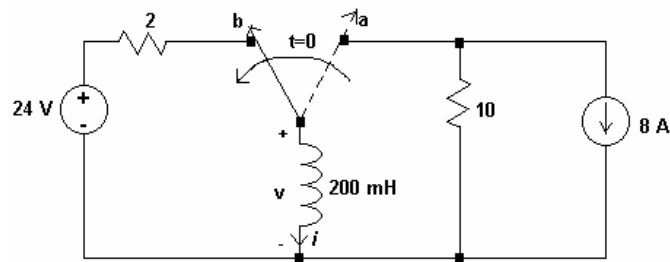
(c) Linear first order differential equation:  $L \frac{di}{dt} + Ri = V_s \rightarrow \frac{di}{dt} + \frac{R}{L}i = \frac{V_s}{L}$

(d) Then  $\tau = \frac{L}{R}$  and  $A = \frac{V_s}{L}$ ;  $i(\infty) = \tau \cdot A = \frac{V_s}{R}$

(e) Solution for current:  $i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau} = \frac{V_s}{R} + \left\{ i(0) - \frac{V_s}{R} \right\} \cdot e^{-\frac{R}{L}t}$

(f) Solution for voltage across the inductor:  $v_L(t) = L \frac{di(t)}{dt}$

H.2. Example: The switch has been in position  $a$  for a long time. At  $t=0$ , it moves to position  $b$ . Find the voltage across the inductor and the current through it for  $t > 0$ .



SOLUTION:



## I. RC Natural Response Example

### I.1. Summary

(a) Circuit formation: RC Parallel circuit with initial current in the inductor.

No source after  $t > 0$ .

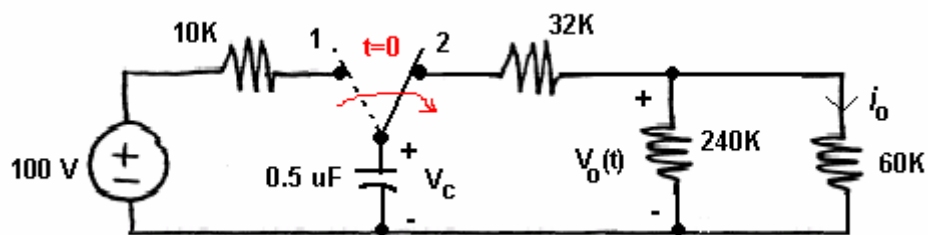
(b) First order differential equation:  $C \frac{dv}{dt} + \frac{v}{R} = 0 \rightarrow \frac{dv}{dt} + \frac{v}{RC} = 0$  with  $\tau = RC$  and  $v(\infty) = 0$

(c) Solution form:  $v(t) = v(0)e^{-\frac{t}{\tau}} = v(0) \cdot e^{-\frac{1}{RC}t}$ ,

(d) Power equation:  $p = vi$

(e) Energy equation:  $w = \int_0^t p dx$

I.2 Example: The switch has been in position 1 for a long time. At  $t=0$ , it is switched to position 2. Find (a)  $v_c(t)$ ,  $v_o(t)$ , and  $i_o(t)$ ; (b) total energy dissipated in 60K resistor.



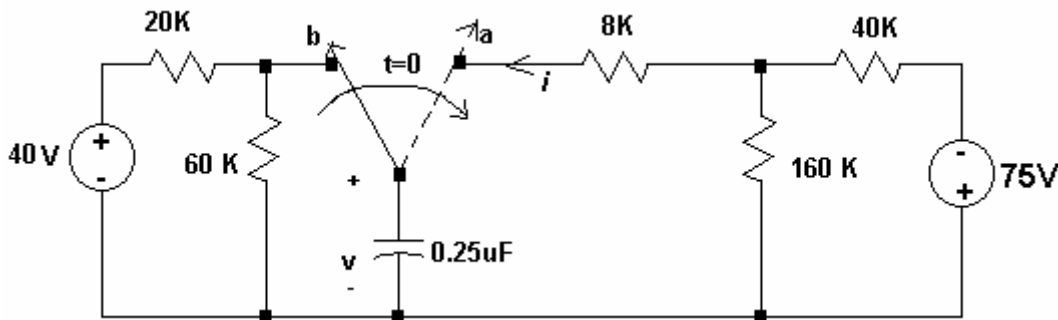
SOLUTION:

## J. RC Step Response Example

### J.1. Summary

- (a) Circuit formation: R-C Parallel. No source before  $t=0$ .
- (b) DC current source ( $I_s$ ) is suddenly applied at  $t=0$ .
- (c) Linear first order differential equation:  $C \frac{dv}{dt} + \frac{v}{R} = I_s \rightarrow \frac{dv}{dt} + \frac{v}{RC} = \frac{I_s}{C}$
- (d)  $\tau=RC$  and  $A = \frac{I_s}{C}$ ;  $v(\infty) = \tau \cdot A = I_s R$
- (f) Solution for voltage:  $v_c(t) = v_c(\infty) + [v_c(0) - v_c(\infty)] \cdot e^{-\frac{t}{RC}}$ ,  $\tau = RC$
- (g) Solution for current:  $i_c(t) = C \frac{dv_c(t)}{dt}$

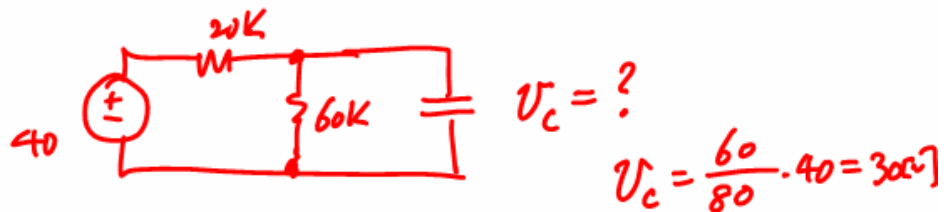
J.2. Example: The switch has been in position  $b$  for a long time. At  $t=0$ , it moves to position  $a$ . Find the voltage across the capacitor and the current through it for  $t>0$ .



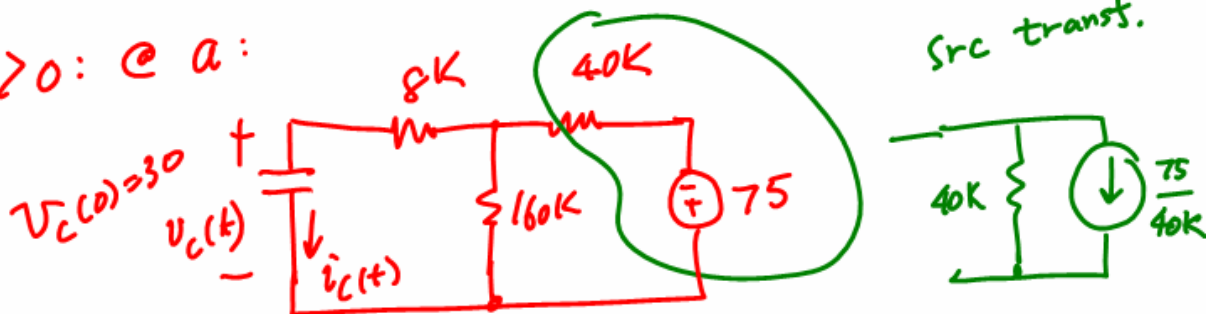
SOLUTION:

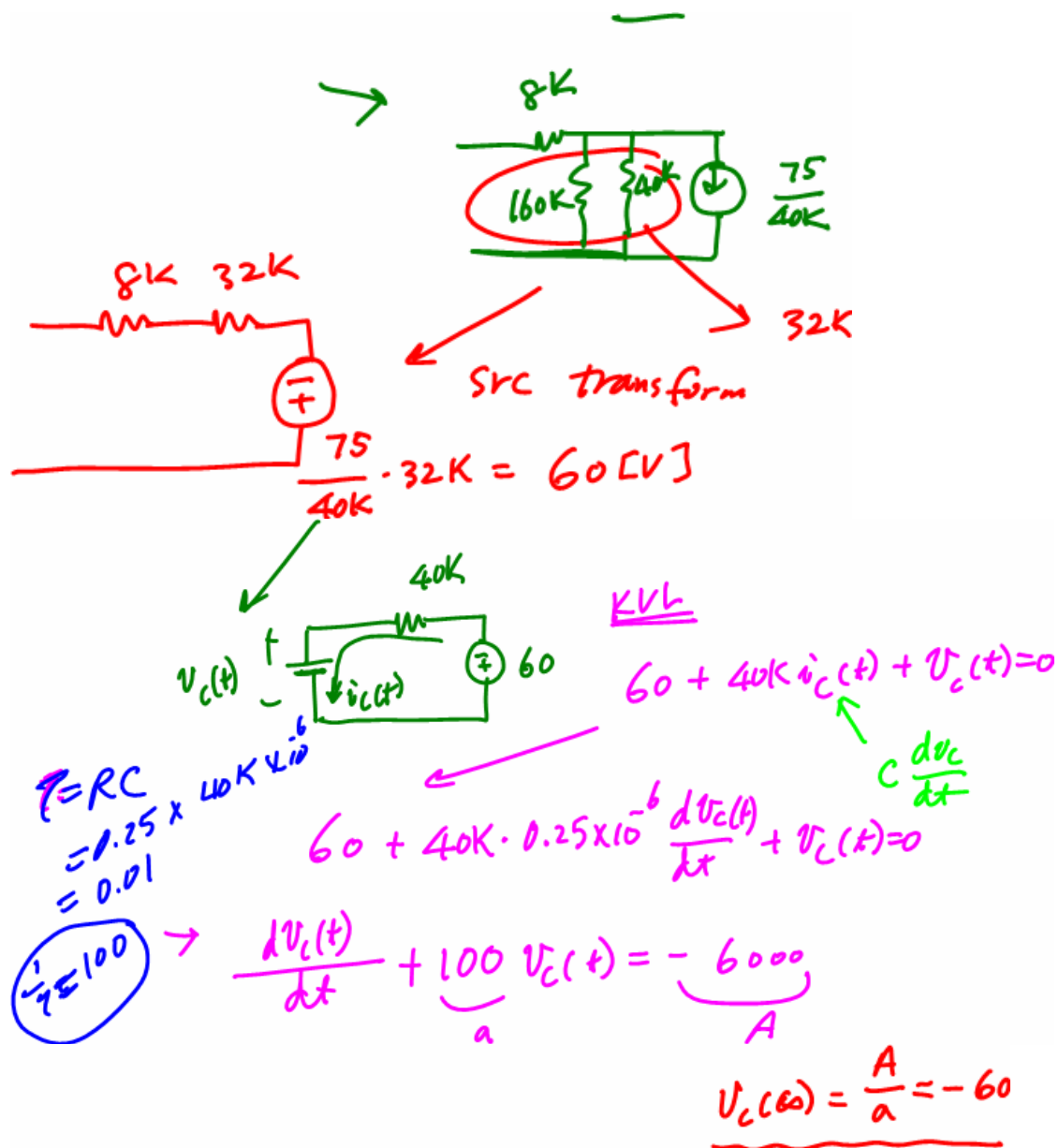
Sol)

$t < 0$ : @  $b$ :



$t > 0$ : @  $a$ :





$$\begin{aligned}
 v_c(t) &= v_c(\infty) + [v_c(0) - v_c(\infty)] e^{-100t} \\
 &= -60 + [30 - (-60)] e^{-100t} \\
 &= -60 + 90 e^{-100t}
 \end{aligned}$$

$$\begin{aligned}
 i_c(t) &= C \frac{dv_c(t)}{dt} = (0.25 \times 10^{-6}) [-60 + 90 e^{-100t}]' \\
 &= (0.25 \times 10^{-6}) (-9000 e^{-100t}) \\
 &= -2.25 \times 10^{-3} e^{-100t} \text{ [A]}
 \end{aligned}$$