Class Note 21: Inductors and Capacitors

A. Inductor

1. An inductor is a passive element designed to store energy in its magnetic field.
2. A practical inductor is usually formed into a cylindrical coil with many turns of conducting wires.

3. The voltage across an inductor is directly proportional to the time rate of change of the current through the inductor: $v(t) = L\frac{di(t)}{dt}$, where L is the constant of proportionality called the *inductance* of the inductor, which is the property whereby an inductor exhibits ***opposition to the changes of current flowing*** through it.
4. The unit of inductance is the henry (H), named in honor of the American inventor Joseph Henry (1797-1878). 1 henry equals 1 volt-second per ampere.
5. The current-voltage relationship is: $i(t) = \frac{1}{L}\int_{t_0}^{t} v(y)dy + i(t_0)$
6. The energy stored in an inductor, since the power delivered to an inductor is $p = vi = L\frac{di}{dt}\cdot i$, can be: $w = \int_{-\infty}^{t} p d\tau = \int_{-\infty}^{t} (L\frac{di}{dt})i d\tau = L\int_{-\infty}^{t} i di = \frac{1}{2}Li^2(t) - \frac{1}{2}Li^2(-\infty) = \frac{1}{2}Li^2$
7. Important properties of an inductor.
   (a) Note that, from $v(t) = L\frac{di(t)}{dt}$, the voltage across an inductor is zero when the current is constant. ----> An inductor acts like a short circuit to DC.
   (b) Note that, from $i(t) = \frac{1}{L}\int_{t_0}^{t} v(y)dy + i(t_0)$, the current through an inductor cannot change instantaneously.
   (c) Note that, however, from $v(t) = L\frac{di(t)}{dt}$, the voltage across an inductor can change abruptly.
   (d) An ideal inductor does not dissipate: the energy stored in it can be retrieved at a later time. The inductor takes power from the circuit when storing energy and delivers power to the circuit when returning previously stored energy.

8. The equivalent inductance of series-connected inductors is the sum of the individual inductances: $L_{eq} = \sum_{k=1}^{n} L_k$
9. The equivalent inductance of parallel inductors is the reciprocal of the sum of the reciprocals of the individual inductances: $\frac{1}{L_{eq}} = \sum_{k=1}^{n} \frac{1}{L_k}$

B. Capacitors

1. A capacitor is a passive element designed to stored energy in its electric field.
2. A capacitor consists of two conducting plates separated by an insulator (or dielectric). In many applications, the plates may be aluminum foil while the dielectric may be air, ceramic, paper, or mica.
3. Commercially available capacitors are, by the dielectric materials they are used of, polyester capacitors (light and stable), film capacitors, and electrolytic capacitors (high capacitance).

4. When a voltage source (v) is connected the a capacitor, the amount of charge stored (q) is directly proportional to the applied voltage: $q = Cv$, where C, the constant, is the *capacitance* of the capacitor. In other words, capacitance is the ratio of the charge on one plate of a capacitor to the voltage difference between the two plates.
5. The unit of the capacitance is the farad(F), in honor of the English physicist Michael Faraday (1791-1867). 1 F = 1 Coulomb/Volt.
6. The equation $q = Cv$ can now changed, since $i = \frac{dq}{dt}$, to $i = \frac{dq}{dt} = \frac{d(Cv)}{dt} = C\frac{dv}{dt}$.
7. Voltage-current relationship can be obtained by integrating both sides of $i = C\frac{dv}{dt}$:

$$v(t) = \frac{1}{C}\int_{t_0}^{t} i(x)dx + v(t_0)$$

8. The energy stored in a capacitor, since the instantaneous power delivered to the capacitor is

$p = vi = vC\frac{dv}{dt}$, can be: $w = \int_{-\infty}^{t} p\,d\tau = C\int_{-\infty}^{t} v\frac{dv}{dt}d\tau = L\int_{-\infty}^{t} v\,dv = \frac{1}{2}Cv^2(t) - \frac{1}{2}Cv^2(-\infty) = \frac{1}{2}Cv^2$

9. Important properties of a capacitor

(a) Note that, from $i = C\frac{dv}{dt}$, when the voltage across a capacitor is not changing with time, the current through the capacitor is zero. ----> A capacitor is an open circuit to DC.

(b) Note that, from $v(t) = \frac{1}{C}\int_{t_0}^{t} i(x)dx + v(t_0)$, the voltage on the capacitor cannot change abruptly; instead, the voltage must be continuous.

(c) However, the current through a capacitor can change instantaneously.
(d) An ideal capacitor does not dissipate energy. It takes power from the circuit when storing (or charging) energy and returns previously stored energy when delivering (or discharging) power the circuit.

10. The equivalent capacitance of series-connected capacitors is the reciprocal of the sum of the reciprocals of the individual capacitances: $\frac{1}{C_{eq}} = \sum_{k=1}^{n} \frac{1}{C_k}$
11. The equivalent capacitance of parallel-connected capacitors is the sum of the individual capacitors: $C_{eq} = \sum_{k=1}^{n} C_k$

C. Summary Table for Inductors and Capacitors

	Inductor	Capacitor
Voltage Equation	$v(t) = L\frac{di(t)}{dt}$	$v(t) = \frac{1}{C}\int_{t_0}^{t} i(x)dx + v(t_0)$
Current Equation	$i(t) = \frac{1}{L}\int_{t_0}^{t} v(y)dy + i(t_0)$,	$i(t) = C\frac{dv(t)}{dt}$
Power Equation	$p(t) = v(t) \cdot i(t)$ or $p(t) = L \cdot i(t) \cdot \frac{di(t)}{dt}$ or $p(t) = v(t) \cdot \{\frac{1}{L}\int_{0}^{t} v(y)dy + i(0)\}$	$p(t) = v(t) \cdot i(t)$ or $p(t) = C \cdot v(t) \cdot \frac{dv(t)}{dt}$ or $p(t) = i(t) \cdot \{\frac{1}{C}\int_{0}^{t} i(x)dx + v(0)\}$
Energy Equation	$w = \frac{1}{2}L \cdot [i(t)^2 - i(0)^2]$	$w = \frac{1}{2}C \cdot [v(t)^2 - v(0)^2]$
Series Combination	$L_{eq} = \sum_{k=1}^{n} L_k$	$\frac{1}{C_{eq}} = \sum_{k=1}^{n} \frac{1}{C_k}$
Parallel Combination	$\frac{1}{L_{eq}} = \sum_{k=1}^{n} \frac{1}{L_k}$	$C_{eq} = \sum_{k=1}^{n} C_k$
Behavior at DC	Short Circuit	Open Circuit
Variable that cannot change abruptly	Voltage, v	Current, i

D. The Rest of the Operational Amplifier[1]

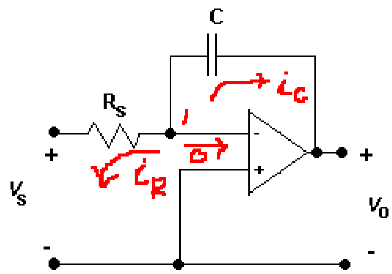
1. In the previous chapter, we discussed about the following op amp circuits: summer and subtractor.

[1] This subject is further discussed in ELEG 301 Network Analysis II.

2. We will discuss two more op amp circuits that had been widely used in analog computers: integrator and differentiator.

3. An **integrator** is an op amp circuit whose output is proportional to the integral of the input signal.

   (a) Consider a circuit below. This is the familiar inverting amplifier circuit, replacing the feedback resistor by a capacitor.



   (b) A node-voltage equation at node 1: $i_R + i_C = 0$, where $i_R = \frac{0 - v_s}{R_s}$ and $i_C = -C\frac{dv_o}{dt}$.

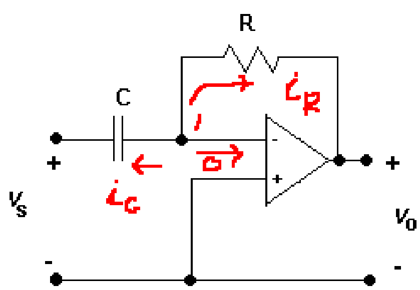
   (c) Therefore, the current equation becomes: $\frac{v_s}{R_s} = -C\frac{dv_o}{dt}$ ----> $dv_o = -\frac{1}{RC}v_s dt$

   (d) Integrating both sides gives $v_o(t) - v_o(0) = -\frac{1}{RC}\int_0^t v_s(t)dt$

   (e) Assuming $v_o(0)=0$ (discharging the capacitor prior to the application of the input signal), we have $v_o(t) = -\frac{1}{RC}\int_0^t v_s(t)dt$.

4. A **differentiator** is an op amp circuit whose output is proportional to the rate of change of the input signal.

   (a) Consider another circuit shown below.



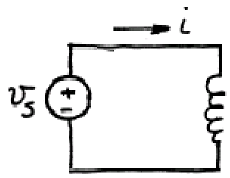
   (b) Applying KCL at node 1: $i_R + i_C = 0$ $i_R = \frac{0 - v_o}{R}$ and $i_C = -C\frac{dv_s}{dt}$.

   (c) Therefore, we have: $\frac{v_o}{R} = -C\frac{dv_s}{dt}$ ----> $v_o(t) = -RC\frac{dv_s(t)}{dt}$

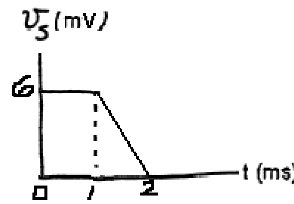
   (d) Caveat: Differentiator circuits are electronically unstable because any electrical noise within the circuit is exaggerated by the differentiator. Hence, the differentiator circuit is not as useful and popular as the integrator. It is seldom used in practice.

E. Example Problems

1. The voltage at the terminals of the 300 uH inductor of the circuit (a) is shown in (b). The inductor current i is known to be zero before time t=0. Derive the expression for i (for t>0) and sketch it.



a



b

SOLUTION:

$0 \le t \le 1$ ms :

$$i = \frac{1}{L}\int_0^t v_s \, dx + i(0) = \frac{10^6}{300}\int_0^t 6 \times 10^{-3} \, dx + 0$$

$$= 20x \Big|_0^t = 20t \text{ mA}$$

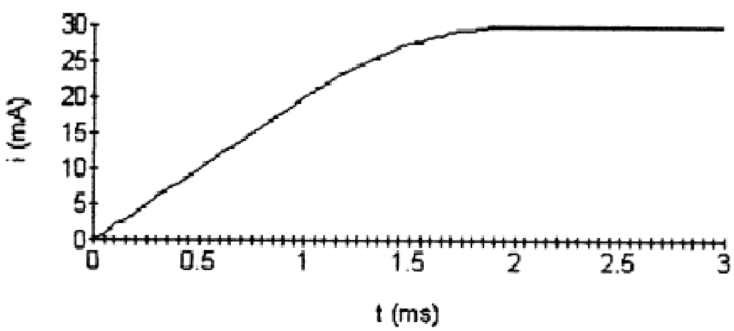
$1 \text{ ms} \le t \le 2 \text{ ms}$:

$$i = \frac{10^6}{300}\int_{10^{-3}}^t (12 \times 10^{-3} - 6x) \, dx + 20 \times 10^{-3}$$

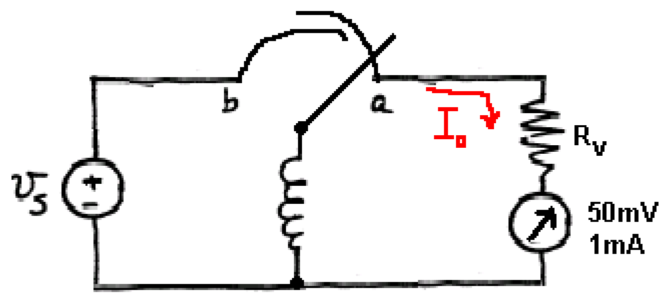
$$\therefore \quad i = 40t - 10{,}000t^2 - 10 \times 10^{-3} \text{ mA}$$

$2 \text{ ms} \le t \le \infty$:

$$i = \frac{10^6}{300}\int_{2\times10^{-3}}^t (0) \, dx + 30 \times 10^{-3} = 30 \text{ mA}$$



2. Initially there was no energy stored in the 25 H inductor when it was placed across the terminals of the voltmeter (with full-scale of 50 V). At t=0, the inductor was switched instantaneously to position ***b*** where it remained for 1 second before returning instantaneously to position ***a***. What will be the reading of the voltmeter be at the instant the switch returns to position ***a***? The d'Arsonval movement has the rating of 50mV@ 1mA. Note that V_s=20 [mV].



SOLUTION:

(a) R_m=50 and R_v=49950 --->R_{eq}=50 k

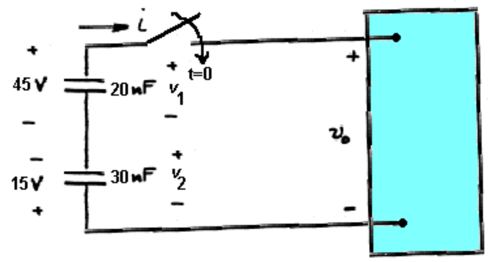
(b)

$$I_o = \frac{1}{25}\int_0^1 20 \times 10^{-3}\, dt = 0.8t \Big|_0^1 = 0.8\,\text{mA}$$

(c) Therefore, reading is: $50 \cdot \frac{0.8}{1} = 40$ [V]

3. The two series-connected capacitors are connected to the terminals of a black box at t=0. The resulting current i(t) for t>0 is known to be $i(t) = 900e^{-2500t}$ [uA]

(a) How much energy was initially stored in the series capacitors?
(b) Find v_1(t) for t>0
(c) Find v_2(t) for t>0
(d) find v(t) for t>0
(e) How much energy is delivered to the black box in the time interval 0<t<∞?



SOLUTION:

(a) From $w = \frac{1}{2}C \cdot [v(t)^2 - v(0)^2]$ and v(t)=0 at t=0

$$w = \tfrac{1}{2}(20 \times 10^{-9})(45)^2 + \tfrac{1}{2}(30 \times 10^{-9})(15)^2 = 20.25 \times 10^{-6} + 3.375 \times 10^{-6}$$
$$= 23.625\,\mu J$$

(b) and (c)

$$v_1 = -\frac{10^9}{20}(900 \times 10^{-6})\frac{e^{-2500x}}{-2500}\Big|_0^t + 45 \qquad v_2 = -\frac{10^9}{30}(900 \times 10^{-6})\frac{e^{-2500x}}{-2500}\Big|_0^t - 15$$

$$= 18e^{-2500t} + 27\,V, \quad t \geq 0 \qquad = 12e^{-2500t} - 27\,V, \quad t \geq 0$$

(d) C=[20*30]/(20+30)=12 nF

$$v_o = -\frac{10^9}{12}\int_0^t 900 \times 10^{-6} e^{-2500x}\,dx + 30 = -75{,}000\frac{e^{-2500x}}{-2500}\Big|_0^t + 30 = 30e^{-2500t}\,V, \quad t \geq 0$$

(e)

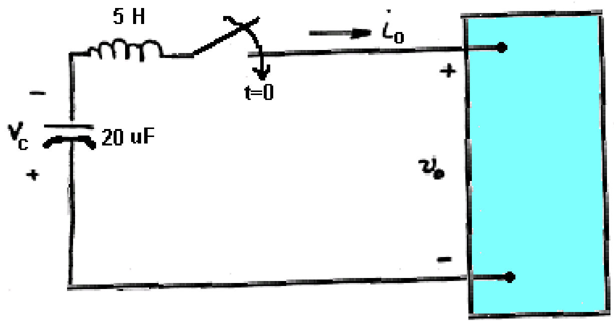
$$w]_0^\infty = w(\infty) - w(0) = \{\frac{1}{2}C \cdot [v(\infty)^2 - v(0)^2]\} - \{\frac{1}{2}C \cdot [v(0)^2 - v(0)^2]\}$$
$$= \{\frac{1}{2}C \cdot [v(\infty)^2 - v(0)^2]\}$$

Since v(∞)=0 (see the equation for v(t)) and v(0)=30 V

$$w]_0^\infty = w(\infty) - w(0) = \{\frac{1}{2}C \cdot [v(\infty)^2 - v(0)^2]\} = \frac{1}{2}C \cdot [0 - v(0)^2] = -5.4\ [\text{uJ}]$$

Therefore 5.4 uJ is delivered to the box.

4. At t = 0, a series-connected capacitor and inductor are placed across the terminals of a black box. For t>0, it is know that $i_o(t) = e^{-80t}\sin 60t$ [A]. Find v_0(t) for t>0, if $v_c(0)$= 300 [V]



NOTE: Integration formula: $\int e^{ax}\cos(bx)dx = \frac{e^{ax}}{a^2+b^2}[a\cos(bx)+b\sin(bx)]$

$\int e^{ax}\sin(bx)dx = \frac{e^{ax}}{a^2+b^2}[a\sin(bx)-b\cos(bx)]$

SOLUTION:

$$v_c = \frac{-10^6}{20}\int_0^t e^{-80x}\sin 60x\,dx + 300$$

$$= 5e^{-80t}[80\sin 60t + 60\cos 60t] - 300 + 300$$

$$= 400e^{-80t}\sin 60t + 300e^{-80t}\cos 60t \text{ V}$$

$$v_L = 5\frac{di_o}{dt} = 5[-80e^{-80t}\sin 60t + 60e^{-80t}\cos 60t]$$

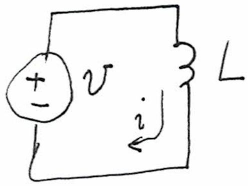
$$= -400e^{-80t}\sin 60t + 300e^{-80t}\cos 60t \text{ V}$$

$$v_o = v_c - v_L$$

$$= (300e^{-80t}\cos 60t - 300e^{-80t}\cos 60t + 400e^{-80t}\sin 60t + 400e^{-80t}\sin 60t)$$

$$= 800e^{-80t}\sin 60t \text{ V}$$

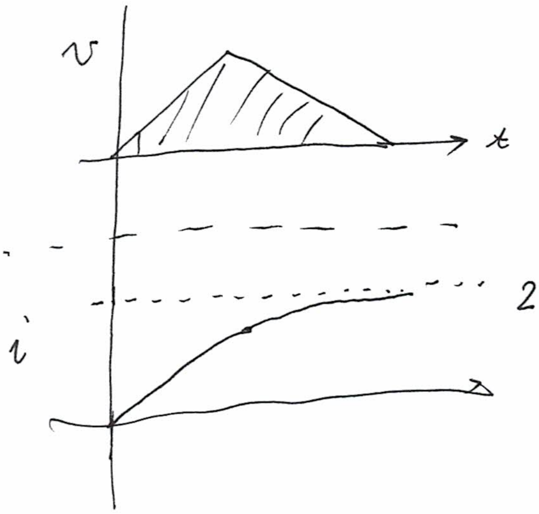
L & C Inductor/Capacitor



$$\boxed{v = L\frac{di}{dt}} \rightarrow v\,dt = L\,di$$

$$\int_0^t v\,d\tau = L\int_{i(0)}^{i(t)} dj = L\left[j\right]_{i(0)}^{i(t)} = L\left[i(t) - i(0)\right]$$

$$\boxed{i(t) = \frac{1}{L}\int_0^t v\,d\tau + i(0)}$$



(EX) $v(t) = 20te^{-10t}$ [V]
with $i(0) = 0$, $L = 0.1H$

$$\rightarrow i(t) = \frac{1}{L}\int_0^t 20\tau e^{-10\tau}\,d\tau$$

$(xy)' = x'y + xy'$

$\tau\left(-\frac{1}{10}e^{-10\tau}\right)'$

$$= 200\int_0^t \tau e^{-10\tau}\,d\tau =$$

$$= 200\left\{\left[\tau\left(-\tfrac{1}{10}\right)e^{-10\tau}\right]_0^t - \left(-\tfrac{1}{10}\right)\int_0^t e^{-10\tau}\,d\tau\right\}$$

$$= 200\left[-\frac{t}{10}e^{-10t} + \frac{1}{10}\cdot\left[-\frac{e^{-\tau}}{10}\right]_0^t\right]$$

$$= 200\left[-\frac{t\cdot e^{-10t}}{10} - \frac{e^{-10t}}{100} + \frac{1}{100}\right]$$

$$= 2\left[1 - 10\cdot t\cdot e^{-10t} - e^{-10t}\right]$$

Check $i(\infty) = 2$

$p = vi$
$= L\frac{di}{dt}\cdot i$

Energy (ω) $p = \frac{d\omega}{dt} = Li\frac{di}{dt}$

$$\rightarrow d\omega = Li\,di$$

$$\int_0^\omega dx = L\int_{i(0)}^{i} j\,dj$$

$$x\Big]_0^\omega = L\frac{j^2}{2}\Big]_0^i$$

$$\boxed{\omega = \frac{1}{2}Li^2}$$

$$\rightarrow \omega = \frac{1}{2}L\cdot 2^2$$
$$= \frac{1}{2}(0.1)(4) = 0.2\ [w]$$

Capacitance

$i = C\frac{dv}{dt}$ → $i\,dt = C\,dv$ →

$\int_0^t i\,d\tau = C\int_{v(0)}^{v(t)} dy = C[y]_{v(0)}^{v(t)} = C[v(t) - v(0)]$

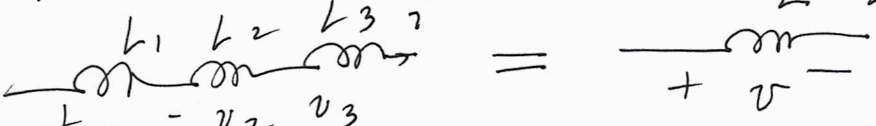
→ $v(t) = \frac{1}{C}\int_0^t i\,d\tau + v(0)$

$p = vi = Cv\frac{dv}{dt}$

$\frac{dw}{dt} = p = Cv\frac{dv}{dt}$ → $dw = Cv\,dv$

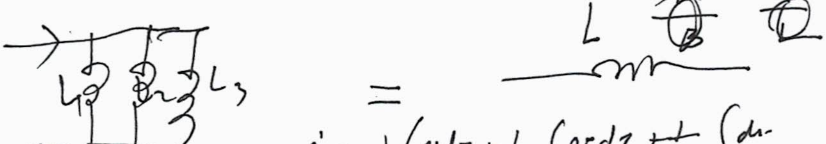
$\int_0^w x = C\int_0^v y\,dy$ ⇒ $\boxed{w = \frac{1}{2}Cv^2}$

Series // Parallel



$L = L_1 + L_2 + L_3$

$v_1 = L_1\frac{di}{dt},\ v_2 = L_2\frac{di}{dt},\ v_3 = L_3\frac{di}{dt}$ → $v = v_1 + v_2 + v_3 = (L_1 + L_2 + L_3)\frac{di}{dt}$



$\frac{1}{L} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}$

$i = \frac{1}{L_1}\int v\,d\tau + \frac{1}{L_2}\int v\,d\tau + \frac{1}{L_3}\int dv$

$= \left(\frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}\right)\int dv = \frac{1}{L}\,dv$

$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$

$C = C_1 + C_2 + C_3$

$C_1\frac{dv}{dt} + C_2\frac{dv}{dt} + ~~C_3~~\frac{dv}{dt} = C\frac{dv}{dt}$

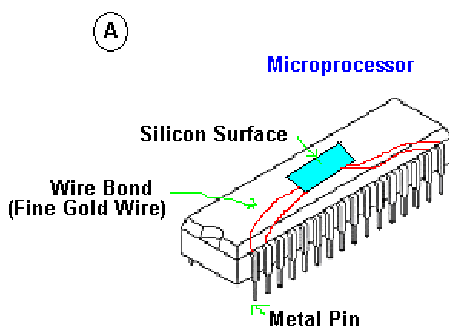
$C = C_1 + C_2 + C_3$

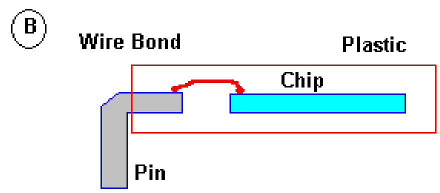
Note21-A: Inductor Note Supplement

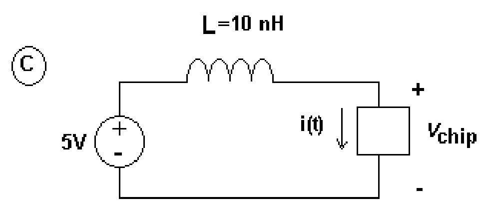
Practical Problem

I. Background

1. ICs are rectangular pieces of silicon.
2. Electrical contact between the silicon and metals pins are made with fine gold wire, called wire bond (Fig. A).
3. The chip is then coated in plastic to protect from physical damage. (Fig. B)
4. Since wires are not perfect conductors, they have resistance and inductance.
5. In most cases, wire resistance and inductance are negligibly small.
6. However, the current (being used by the chip (or processor)) changes quickly, wire inductance can play a significant role.

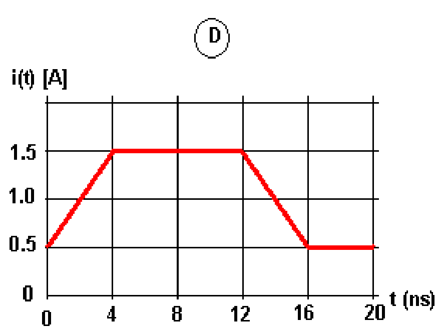


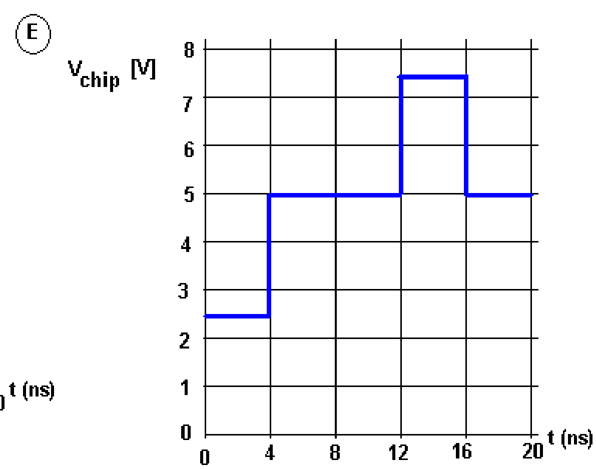




II. Analysis

1. We now examine the influence of the small wire inductance to the voltage across a high speed microprocessor. (See Fig. C for an equivalent circuit with the wire bond modeled by the 10 nH inductor)
2. The chip supply voltage is represented by 5V voltage source.
3. The current i(t) represents the current being used by the microprocessor. And this current demand changes, as the microprocessor executes various functions. An example of current change is shown in Fig. D.





4. Then, voltage across the chip can be expressed by:

$$V_{chip}(t) = 5 - v_L = 5 - L\frac{di(t)}{dt} = 5 - 10 \cdot 10^{-9} \cdot \frac{di(t)}{dt}$$

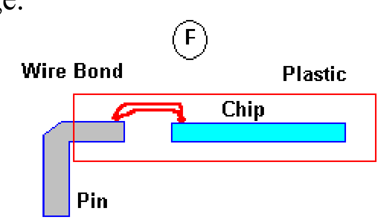
5. $\frac{di(t)}{dt}$ calculation and chip voltage table for 4 time periods:

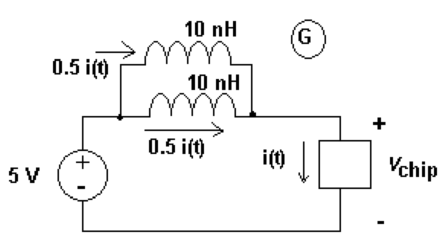
	0 - 4 ns	4 - 12 ns	12 - 16 ns	16 - 20 ns
$\frac{di(t)}{dt} = \frac{\Delta i}{\Delta t}$	$\frac{1.5 - 0.5}{4} = 2.5 \times 10^8$	0	$\frac{0.5 - 1.5}{4} = -2.5 \times 10^8$	0
$L \cdot \frac{di(t)}{dt}$	2.5	0	-2.5	0
$V_{chip}(t)$	2.5	5.0	7.5	0

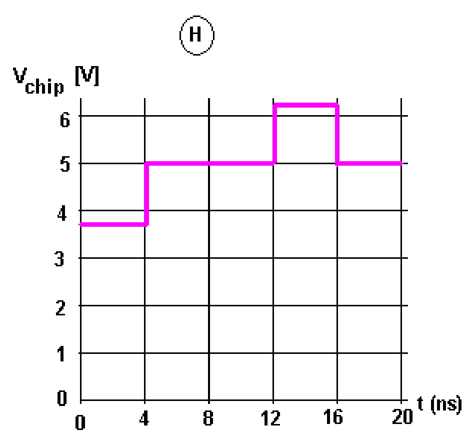
6. The resulting chip voltage is illustrated in Fig.E. As we see, the voltage swings much and the chip voltage is not stable at all.
7. Then, how can we have a more stable chip voltage?
8. The answer comes from the chip voltage equation. If we reduce the inductance *L*, then the sudden voltage shot or drop would be reduced.
9. Let's add one more wire bond between the chip and the metal pin. (See Fig. F)
10. See Fig. G. for a new equivalent circuit.
11. Then, the current will be equally divided in to two inductors.
12. $\frac{di(t)}{dt}$ calculation and chip voltage table for 4 time periods:

	0 - 4 ns	4 - 12 ns	12 - 16 ns	16 - 20 ns
$\frac{di(t)}{dt} = \frac{\Delta i}{\Delta t}$	$\frac{0.75 - 0.25}{4} = 1.25 \times 10^8$	0	$\frac{0.25 - 0.75}{4} = -1.25 \times 10^8$	0
$L \cdot \frac{di(t)}{dt}$	1.25	0	-1.25	0
$V_{chip}(t)$	3.75	5.0	6.25	0

13. The resulting chip voltage is illustrated in Fig.H. As we see, the voltage swings less and the chip voltage is more stable. By adding more wire bonds, we could further stabilize the chip voltage.







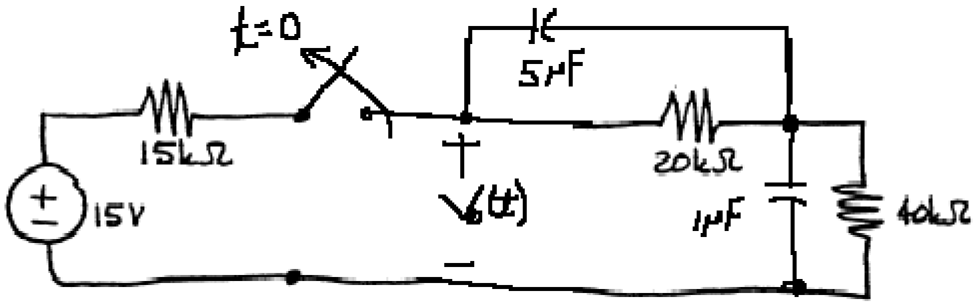
Class Note 23: Transient Circuits

A. Transient Circuits

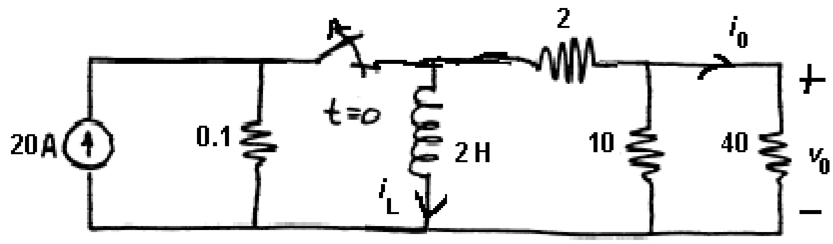
1. We now consider circuits that are in transition from one state to another.
2. The state transition occurs when we suddenly apply to, or instantly remove from, a circuit the source of energy.
3. The analysis of the circuit behavior in the transition phase is called a transient analysis.
4. The transient behavior of circuits is affected by the presence of capacitor or inductor, or both, since these two elements can store energy and releasing it over some interval of time.
5. A circuit comprising a resistor and a capacitor ("RC circuit"), and a circuit comprising a resistor and an inductor ("RL circuit"), result in a first order differential equation.
6. A first order circuit is characterized by a first order differential equation.
7. Both the node equation for a parallel RLC circuit and the mesh equation for a series RLC circuit result in a second-order differential equation.
8. A second order circuit is characterized by a second order differential equation.
9. There are two ways to excite a circuit.
   (a) By initial conditions of the storage elements in the circuit: in this *source-free circuit*, we assume that energy is initially stored in the capacitive or inductive elements of the circuit and the energy causes current to flow in the circuit and is gradually dissipated in the resistors.
   (b) By independent sources: Only DC sources are considered in the course.
10. The natural response of a circuit refers to the behavior (voltages and currents. What else?) of the circuit itself, with no external sources of excitation.
11. The step response of a circuit is its behavior when the excitation is the step function, which may be a voltage or a current source.
12. The two types of first-order circuits and the two ways of exciting them add up to the four possible situations:
   (a) Natural response of RC circuit
   (b) Natural response of RL circuit
   (c) Step response of RC circuit
   (d) Step response of RL circuit

B. First Order Transient Situation (Example Circuits)

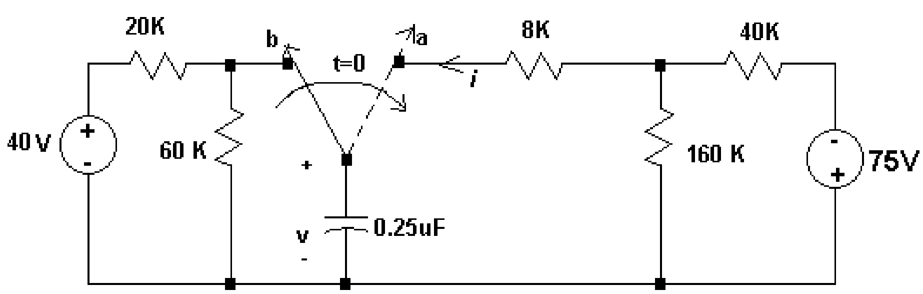
1.Natural response of RC circuit



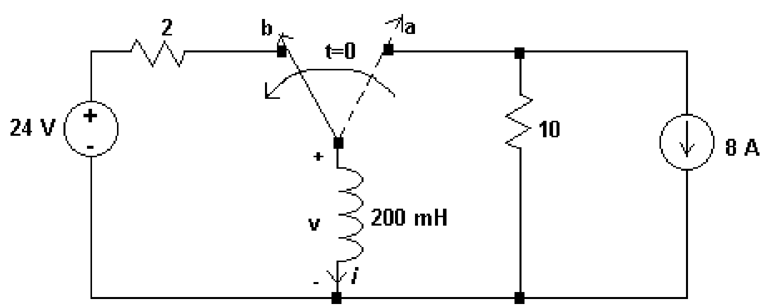
2. Natural response of RL circuit



3. Step response of RC circuit

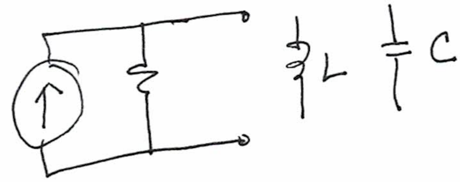


4. Step response of RL circuit



First Order   4 Cases

1) Thevenin $\langle$ L, C   Norton $\langle$ L, C



2) Steady-state current (L) & Voltage (V)

L → $v = L\frac{di}{dt} = 0$ Short circuit

$i = C\frac{dv}{dt} \to 0$ open circuit Voltage

3) Natural Response

$L\frac{di}{dt} + Ri = 0$

$\frac{di}{dt} = -\frac{R}{L} i \;\rightarrow\; di = -\frac{R}{L} i\, dt \;\rightarrow\;$ ~~$\int_{i(0)}^{i(t)} x\, dx = -\frac{R}{L}\int_0^t i\, d\tau$~~

$\frac{di}{i} = -\frac{R}{L} dt$

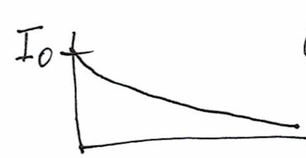
$\frac{1}{i} di = -\frac{R}{L} dt \;\rightarrow\; \int_{i(0)}^{i(t)} \frac{1}{x} dx = -\frac{R}{L}\int_0^t d\tau = -\frac{R}{L} t$

$\rightarrow \ln i(t) - \ln(i(0)) = -\frac{R}{L} t$

$\ln\frac{i(t)}{i(0)} = -\frac{R}{L} t = \ln e^{-\frac{R}{L}t} \;\rightarrow\; i(t) = i(0)\, e^{-(R/L)t}$

$= I_0 e^{-\frac{R}{L}t} = I_0 e^{-t/\tau}$

$i(0) = I_0$, $i(\infty) = 0$



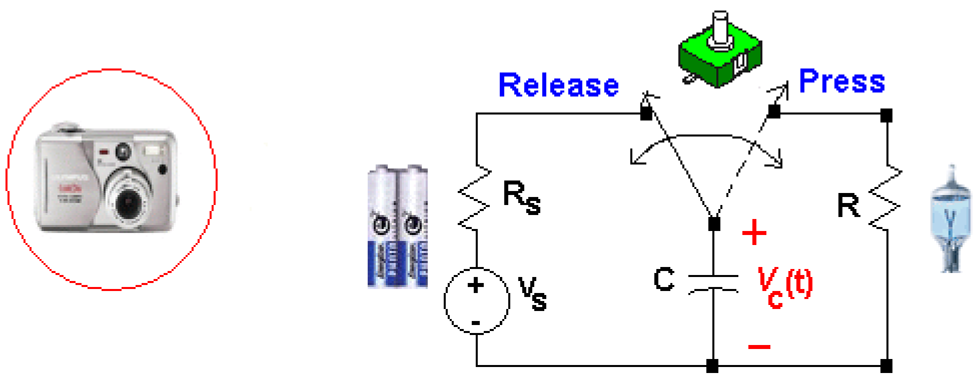
(τ) time const

$\tau = \frac{L}{R}$

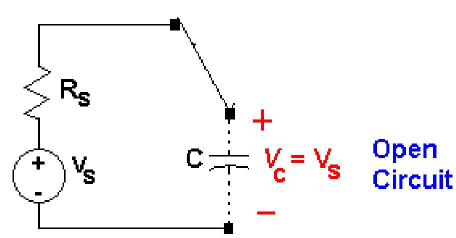
Class Note 24: General solution of a first-order differential equation

A. Example First

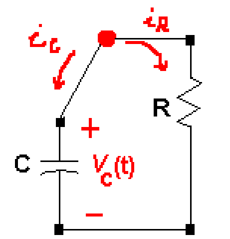
1. Before we solve a first order differential equation, let's consider an example circuit.
2. Consider the flash circuit in a camera. The operation of the flash circuit, from a user standpoint, involves depressing the push button on the camera that triggers both the shutter and the flash and then waiting a few seconds before repeating the process to take the next picture.
3. This operation can be modeled using the circuit below. The voltage source V_s and the resistor R_s model the battery that power the camera and flash. The capacitor models the energy storage, the switch models the push button. And the resistor R models the xenon flash lamp.



4. The capacitor is charged when push button is in the released position.
5. When the button is pressed, the capacitor energy is released through the xenon lamp, producing the flash. In practice, this energy release takes about 1 ms and the discharge time is a function of the elements in the circuit.
6. When the button is released, the battery recharges the capacitor. Again, the time required to charge the capacitor is a function of the circuit elements.
7. Let's further investigate the charging and discharging of the capacitor:
   (a) Charging the capacitor (push button in released position): In DC circuit, the capacitor is an open circuit, therefore, no current flows through the circuit. Hence, the voltage across the capacitor is same as the source voltage.



(b) Discharging the capacitor (push button pressed): when the switch is closed, the node voltage equation, from $i_C + i_R = 0$, is: $C\dfrac{dv_c(t)}{dt} + \dfrac{v_c(t)}{R} = 0$ ---(A.1)



(c) The equation (A.1) is rearranged as: $\dfrac{dv_c(t)}{dt} + \dfrac{1}{RC}v_c(t) = 0$ ----(A.2)

8. If we write the equation (2) using a general variable $x(t)$, instead of the voltage variable, $v_c(t)$, we could have the following a general first order differential equation, which is the subject of the next section. $\dfrac{dx(t)}{dt} + ax(t) = 0$ -----(A.3)

B. General solution of a first-order differential equation

1. Let's start from a first order differential equation of the form

$$\frac{dx(t)}{dt} + ax(t) = A \text{ , (} A \text{ is some constant)----(B.1)}$$

*: Note that a and A in (B.1) correspond to $\dfrac{1}{RC}$ and 0, respectively, in (A.2)

2. A fundamental theorem of differential equation states that:

IF

$x(t) = x_p(t)$ is <u>any solution</u> of equation $\dfrac{dx(t)}{dt} + ax(t) = A$ ---(B.1)

AND IF

$x(t) = x_c(t)$ is <u>any solution</u> to the homogeneous equation $\dfrac{dx(t)}{dt} + ax(t) = 0$ ---(B.2)

THEN

$x(t) = x_p(t) + x_c(t)$ -----(B.3) is <u>a solution</u> to the original equation (B.1)

3.The term $x_p(t)$ is called the <u>particular integral solution</u> (or <u>forced response</u>), and the term $x_c(t)$ is called the <u>complementary solution</u> (or <u>natural response</u>).

4. Then, the general solution of the equation (B.1) consists of two parts that are obtained by solving the two equations:

$$\frac{dx_p(t)}{dt} + ax_p(t) = A \quad \text{----(B.4)}$$

$$\frac{dx_c(t)}{dt} + ax_c(t) = 0 \quad \text{----(B.5)}$$

5. Since the right-hand side of equation (B.4) is a constant, it is reasonable to assume that the solution $x_p(t)$ must also be a constant. Therefore, we assume that: $\underline{x_p(t) = K_1}$ -----(B.6)

6. Substituting this constant in to equation (B.4) yields: $K_1 = \frac{A}{a}$ -----(B.7)

7. Equation (B.5) can be rearranged to: $\frac{dx_c(t)/dt}{x_c(t)} = -a$ ------(B.8)

8. The equation (B.8) is equivalent to: $\frac{d}{dt}[\ln x_c(t)] = -a$

9. Therefore, equation (B.9) becomes: $\ln x_c(t) = -at + c$

10. Equation (B.10), then, becomes: $\ln x_c(t) = \ln[e^{-at+c}] = \ln[e^{-at} \cdot e^c] = \ln[K_2 \cdot e^{-at}]$

11. Therefore, $x_c(t) = K_2 \cdot e^{-at}$ ----(B.9)

12. Finally, the solution to equation (B.1) is:

$$x(t) = K_1 + K_2 e^{-at} \quad \text{----(B.10)}$$

13. The constant K_2 (also K_1) can be found if the value of the independent variable *x*(t) is known at two one instances of time.

    Let's evaluate x(t) at t = t_0: $x(t_0) = K_1 + K_2 e^{-at_o}$

    Also at t=∞: $x(\infty) = K_1 + K_2 e^{-a\infty} = K_1$

    From above two equations, we could get for K_2: $K_2 = K[x(t_0) - x(\infty)]e^{at_0}$

14. Then, we can rewrite the solution (B.10) in to:

$$x(t) = K_1 + K_2 e^{-at} = x(\infty) + [x(t_0) - x(\infty)]e^{-a(t-t_0)} \quad \text{(B.11)}$$

15. If we choose t_0=0, then the solution becomes:

$$x(t) = K_1 + K_2 e^{-at} = x(\infty) + [x(0) - x(\infty)]e^{-at} \quad \text{(B.12)}$$

16. In plain term, we could say this way:

    (Solution)=(Final value) + [(initial value)-(final value)] exp[-*at*]

C. Camera Flash Circuit Case

1. Let's go back to the camera flash circuit to apply the general solution for the first order differential equation.

2. The node-voltage equation across the capacitor is given by: $\frac{dv_c(t)}{dt} + \frac{1}{RC} v_c(t) = 0$

3. We see that: a= 1/RC and A=0 ---- K_1=A/a=0.

4. Therefore, the voltage across the capacitor is in the form $v_c(t) = [v_c(0) - v_c(\infty)]e^{-t/RC}$

5. From the circuit, the initial voltage $v_c(0) = V_s$

6. Also, from the circuit, the final voltage at time t=∞, the voltage will die eventually because there is no voltage source in the circuit but a consuming resistor. So $v_c(\infty) = 0$

6. Finally, $v_c(t) = [V_s - 0]e^{-t/RC} = V_s e^{-t/\tau}$, where time constant τ=RC

general solution

$$\frac{di}{dt} + a i = K$$

$$\frac{di}{dt} = -ai + K$$

$$di = -a\left(i - \frac{K}{a}\right) dt \longrightarrow \int_{i(0)}^{i(t)} \frac{dx}{x - \frac{K}{a}} = -a \int_{t=0}^{t} dy$$

$$\ln\left(x - \frac{K}{a}\right)\Big|_{i(0)}^{i(t)} = -a\, y \Big]_0^t$$

$$\ln \frac{i(t) - \frac{K}{a}}{i(0) - \frac{K}{a}} = -at$$

$$i(t) - \frac{K}{a} = \left(i(0) - \frac{K}{a}\right) e^{-at}$$

$$i(t) = \frac{K}{a} + \left(i(0) - \frac{K}{a}\right) e^{-at}$$

$$i(0) = \frac{K}{a} + \left(i(0) - \frac{K}{a}\right) = \underline{i(0)}$$

$$i(\infty) = \frac{K}{a} + \left(i(0) - \frac{K}{a}\right) \cdot 0 = \frac{K}{a}$$

$$i(t) = \frac{K}{a} + \left(i(0) - i(\infty)\right) e^{-at}$$

($\frac{K}{a} = i(\infty)$)

$$i(t) = i(\infty) + [i(0) - i(\infty)]\, e^{-at}.$$

ELEG202 NETWORK ANALYSIS I Dr. Charles Kim

Class Note 25: Time constant

1. Let's consider a differential equation of: $\frac{dx(t)}{dt}+\frac{1}{\tau}x(t)=A$. Then the solution form is:

$$x(t)=x(\infty)+[x(t_0)-x(\infty)]e^{-(t-t_0)/\tau}$$

2. Assuming that t_0=0 yields to:

$$x(t)=x(\infty)+[x(0)-x(\infty)]e^{-t/\tau}$$

4. Let's further simplify our discussion by assuming that x(∞)=0, then:

$$x(t)=x(0)\cdot e^{-t/\tau}$$

3. The rate of the decay is determined by the constant τ, "**time constant**".

4. Let's find the values of the decaying term *x*(t) at time t=0 and t=τ:

$$x(0)=x(0)\cdot e^{-0/\tau}=x(0)$$
$$x(\tau)=x(0)\cdot e^{-\tau/\tau}=x(0)\cdot e^{-1}$$

5. By comparing the values of the decaying term, we have: $\frac{x(\tau)}{x(0)}=\frac{x(0)e^{-1}}{x(0)}=e^{-1}$

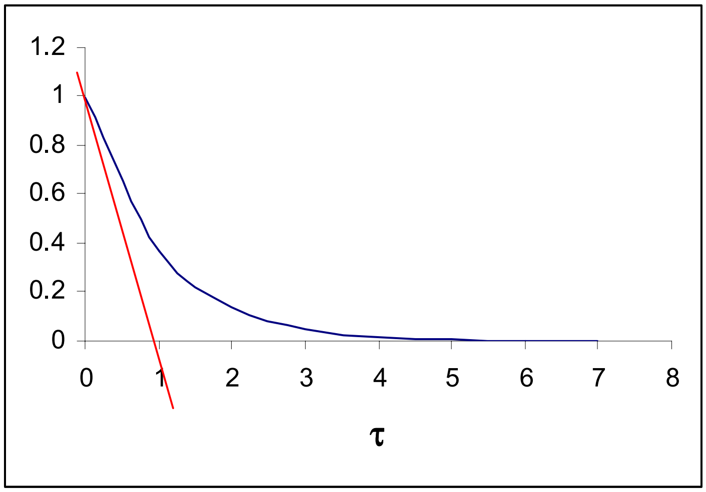
6. The <u>time constant</u> of a circuit then is defined as the time required for the response to decay by a factor of 1/e (or 0.368 or 36.8%) of its initial value.

7. The value of $\frac{x(t)}{x(0)}=\frac{x(0)e^{-t/\tau}}{x(0)}$ for several *t* values in terms of τ is tabulated for a graph:

The value of $\frac{x(t)}{x(0)}=e^{-t/\tau}$	
time *t*=	$e^{-t/\tau}$
0	e^{0}=1.00000
τ	e^{-1}=0.36788
2τ	e^{-2}=0.13534
3τ	e^{-3}=0.04979
4τ	e^{-4}=0.01832
5τ	e^{-5}=0.00674

8. A graph and observations: It is evident that the value *x*(t) is less than 1% of the initial value after 5τ (i.e., five time constants). Thus, it is customary to assume that it takes 5τ for the circuit to reach its final (or steady) state.

9. Another observation: The smaller the time constant, the more rapidly the value *x*(t) decreases.



10. <u>The time constant may be viewed from **another perspective.**</u> Evaluating the derivative of the ratio of x(t) and x(0), we obtain

$$\frac{d}{dt}\left(\frac{x(t)}{x(0)}\right)=\frac{d}{dt}\left(\frac{x(0)e^{-t/\tau}}{x(0)}\right)=\left(-\frac{1}{\tau}e^{-t/\tau}\right)$$

11. The derivative of the ratio at time t = 0, then, becomes $\left(-\frac{1}{\tau}e^{-t/\tau}\right)_{t=0}=-\frac{1}{\tau}$.

12. Thus, the time constant could be defined as "the initial rate of decay," or "the time taken for $\frac{x(t)}{x(0)}$ to decay from unity to zero," assuming a constant rate of decay.

13. In other words, a tangent line, drawn to the decaying curve at t=0, intercepts with the time axis at t = τ. (Refer to the graph above)

ELEG202 NETWORK ANALYSIS I Dr. Charles Kim

Class Note 26: First-Order Circuit Analysis

A. Review

A.1. A general first-order differential equation: $\frac{dx(t)}{dt} + \frac{1}{\tau}x(t) = A$.

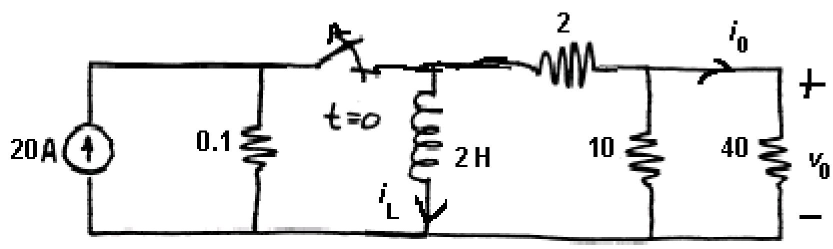
A.2. Then the solution form is: $x(t) = x(\infty) + [x(t_0) - x(\infty)]e^{-(t-t_0)/\tau}$ with $x(\infty) = \tau \cdot A$

A.3. We analyze four categories of first-order circuits. They are
(a)RL Natural Response
(b) RC Natural Response
(c) RL Step Response
(d) RC Step Response

B. RL Natural Response

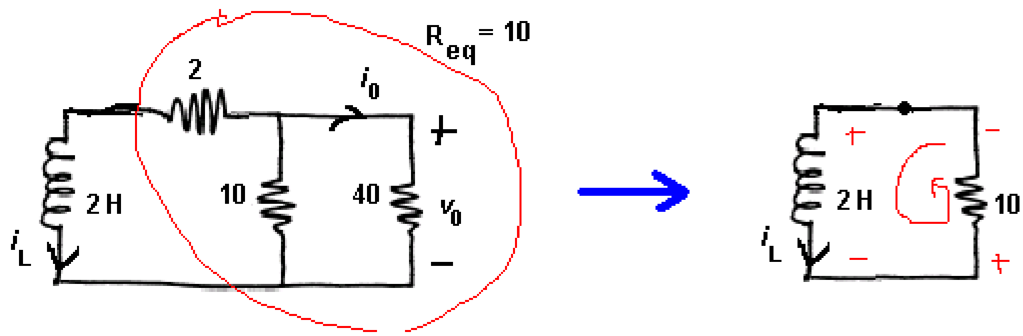
B.1. Summary
(a) Circuit formation: RL Parallel circuit with initial current in the inductor.
No source after t>0.
(b) Linear first order differential equation: $L\frac{di}{dt} + Ri = 0$ --> $\frac{di}{dt} + \frac{R}{L}i = 0$, $x(\infty) = \tau \cdot A$=0
(c) Solution form: $i(t) = i(t_0) \cdot e^{-\frac{t}{\tau}} = i(t_0)e^{-\frac{Rt}{L}}$, $\tau = \frac{L}{R}$
(d) Power equation: $p = vi$
(e) Energy equation: $w = \int_0^t pdx$

B.2. Example #1: The switch has been closed for a long time before being opened at t=0. Find i_L(t), i_0(t), and v_0(t). Also find $P_{10\Omega}$ and $w_{10\Omega}$ for t>0.



SOLUTION:
(a) for t<0: Current source is DC, therefore, the inductor is actually shorted-out. Therefore, all the current from the source flows through the inductor. Therefore, the initial current value of the inductor is 20[A]. That is, $i_L(0)$=20.

(b) for t>0: The total resistance, which is paralleled with the inductor, is 2+10//40=10 Ω



(c) Therefore, this is a RL natural response circuit with L=2H and R=10Ω

(d) Applying KCL yields: $2\frac{di_L}{dt} + 10i_L = 0$ --> $\frac{di_L}{dt} + 5i_L = 0$

(e) Therefore, τ =0.2 and A=0 ---->$i_L(\infty)$=0

(f) The solution form, then, is: $i_L(t) = i(0)e^{-t/\tau} = 20e^{-5t}$

(g) For i_o, we use current-division from i_L(t): $i_0(t) = -i_L(t) \cdot \frac{10}{10+40} = -4e^{-5t}$

(h) Then, $v_0(t) = 40i_0(t) = -160e^{-5t}$

(i) $P_{10\Omega} = \frac{{v_0}^2}{10} = 2560e^{-10t}$ [W]

(j) $w_{10\Omega} = \int_0^\infty 2560e^{-10x}dx = 256\left[0 - e^0\right] = 256$ [J]

(k)Observation:

i. Initial energy stored in the inductor: $w(0) = \frac{1}{2}Li(0)^2 = \frac{1}{2} \cdot 2 \cdot 20^2 = 400$ [J]

b. Where is the difference of (400-256)=144[J]?

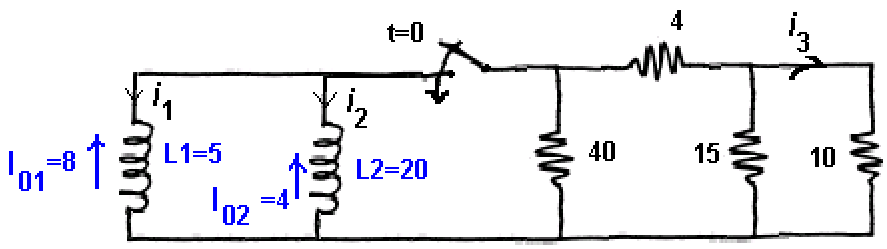
$w_{2\Omega} = ?$

$P_{2\Omega} = 2 \cdot {i_L}^2 = 800e^{-10t}$ ---> $w_{10\Omega} = \int_0^\infty 800e^{-10x}dx$ =80 [J]

$w_{40\Omega}$ =?

$P_{40\Omega} = \frac{{v_o}^2}{40} = 640e^{-10t}$ ---> $w_{40\Omega} = \int_0^\infty 640e^{-10x}dx$ =64 [J]

B3. Example #2: Two inductors, L1 and L2, with initially charged (I_{01}=8 [A] and I_{02}=4 [A]) are connected to a resistive circuit at t=0. Find (a) $i_1(t)$, $i_2(t)$, and $i_3(t)$ for t>0; (b) initial energy stored in L1 and L2; (c) Energy stored in L1 and L2 as t →∞; and (d) Energy delivered to the resistive circuit as t →∞.



SOLUTION:

C. RC Natural Response

C.1. Summary

(a) Circuit formation: RC Parallel circuit with initial current in the inductor.
No source after t>0.

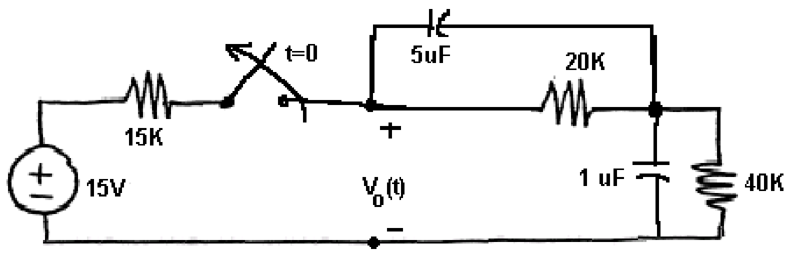
(b) First order differential equation: $C\frac{dv}{dt}+\frac{v}{R}=0$ ---> $\frac{dv}{dt}+\frac{v}{RC}=0$ with $\tau = RC$ and v(∞)=0

(c) Solution form: $v(t)=v(0)e^{-\frac{t}{\tau}}=v(0)\cdot e^{-\frac{1}{RC}t}$,

(d) Power equation: $p=vi$

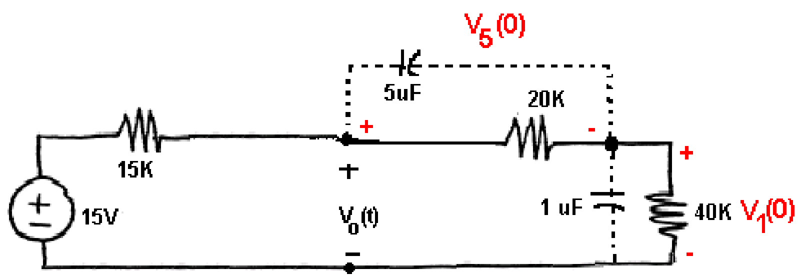
(e) Energy equation: $w=\int_0^t pdx$

C.2 Example #1: The switch has been closed for a long time before being opened at t=0.
Find V_o(t) for t>0.



SOLUTION:

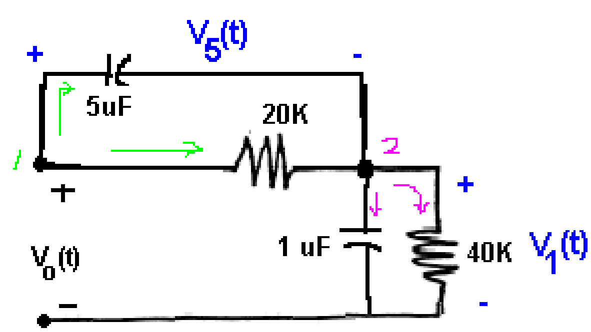
(a) for t<0: Capacitors are open-circuits in DC.



(b) Therefore, the initial voltages across the 5uF and 1uF capacitors are:

$$V_5(0)=15\cdot\frac{20}{15+20+40}=4\text{ [V]} \quad \text{and} \quad V_1(0)=15\cdot\frac{40}{15+20+40}=8\text{ [V]}$$

(c) For t>0: $V_0(t)=V_5(t)+V_1(t)$



(d) @ 1: $C\frac{dv_5}{dt}+\frac{v_5}{R}=0$ ---> $\frac{dv_5}{dt}+\frac{v_5}{RC}=\frac{dv_5}{dt}+\frac{v_5}{0.1}=0$: τ_5 =0.1 and A=0 ----> $v_5(\infty)$=0

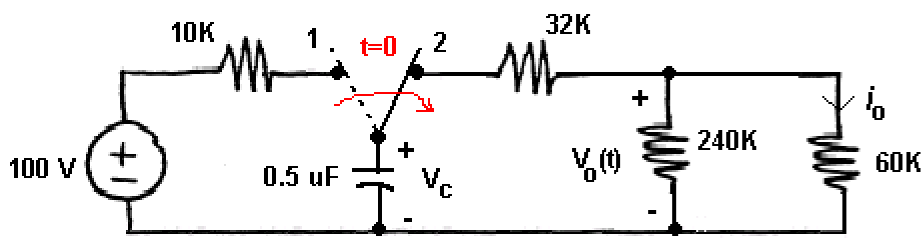
Therefore, $v_5(t)=v_5(0)e^{-\frac{t}{\tau_5}}=4e^{-10t}$

(e) @ 2: $C\frac{dv_1}{dt}+\frac{v_1}{R}=0$ ---> $\frac{dv_1}{dt}+\frac{v_1}{RC}=\frac{dv_1}{dt}+\frac{v_1}{0.04}=0$: τ_1 =0.04 and A=0 ----> $v_1(\infty)$=0

Therefore, $v_1(t)=v_1(0)e^{-\frac{t}{\tau_1}}=8e^{-25t}$

(f) Finally, $v_0(t)=v_5(t)+v_1(t)=4e^{-10t}+8e^{-25t}$

C.3. Example #2: The switch has been in position 1 for a long time. At t=0, it is switched to position 2. Find (a) $v_c(t)$, $v_o(t)$, and $i_o(t)$; (b) total energy dissipated in 60K resistor.



SOLUTION:

D. RL Step Response

D.1. Summary

(a) Circuit formation: R-L Series. No source before t=0.

(b) DC voltage source (V_s) is suddenly applied at t=0.

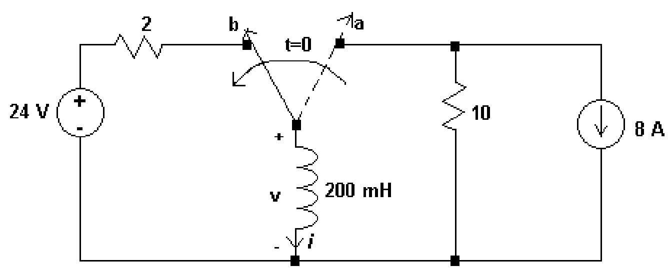
(c) Linear first order differential equation: $L\frac{di}{dt}+Ri=V_s$ ---> $\frac{di}{dt}+\frac{R}{L}i=\frac{V_s}{L}$

(d) Then $\tau=\frac{L}{R}$ and A=$\frac{V_s}{L}$; $i(\infty)=\tau\cdot A=\frac{V_s}{R}$

(e) Solution for current: $i(t)=i(\infty)+[i(0)-i(\infty)]e^{-t/\tau}=\frac{V_s}{R}+\left\{i(0)-\frac{V_s}{R}\right\}\cdot e^{-\frac{R}{L}t}$

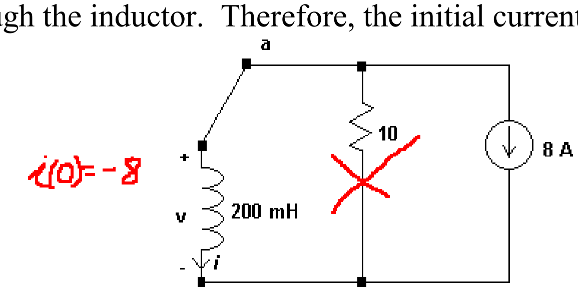
(f) Solution for voltage across the inductor: $v_L(t)=L\frac{di(t)}{dt}$

D.2. Example #1: The switch has been in position *a* for a long time. At t=0, it moves to position *b*. Find the voltage across the inductor and the current through it for t>0.

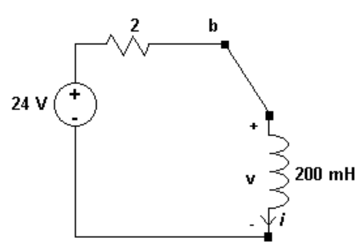


SOLUTION:

(a) for t<0: Inductor is a short circuit in DC, therefore, there is no current through the resistor. All the current flows through the inductor. Therefore, the initial current is $i(0)=-8\,[\text{A}]$



(b) for t>0:



(c) KVL: $-24+0.2\frac{di}{dt}+2i=0$ ---> $\frac{di}{dt}+10i=120$; τ=0.1 and A=120; $i(\infty)=\tau\cdot A=12$

(d) Solution: $i(t)=12+\{-8-12\}\cdot e^{-10t}=12-20\cdot e^{-10t}$

(e) Voltage across the inductor: $v_L(t) = L\frac{di(t)}{dt} = (0.2)\cdot(200)\cdot e^{-10t} = 40e^{-10t}$

(f) Observation:

i. Initial voltage across the inductor

$v_L(0^+)$=40 [V]

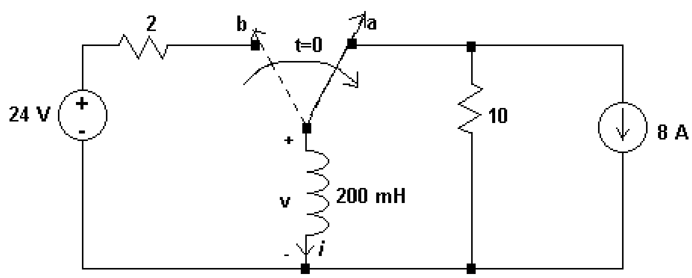
ii. The time at which the inductor voltage equals the source voltage, 24 [V]

$v_L(t) = 40e^{-10t} = 24 \rightarrow -10t = \ln\frac{24}{40} = -0.5108$, therefore t=0.05108 [s]

iii. The voltage across the inductor after a long time.

$v_L(\infty)$=0 [V].

D.3. Example #2: The switch has been in position *b* for a long time. At t=0, it moves to position *a*. Find the voltage across the inductor and the current through it for t>0.



SOLUTION:

(a) for t<0:

(b) for t>0:

(c) KVL equation:

(d)Observation:

i. Initial voltage across the inductor

$v_L(0^+)$=?

ii. The time at which the inductor voltage equals the source voltage, -80 [V]

iii. The voltage across the inductor after a long time.

E. RC Step Response

E.1. Summary

(a) Circuit formation: R-C Parallel. No source before t=0.

(b) DC current source (I_s) is suddenly applied at t=0.

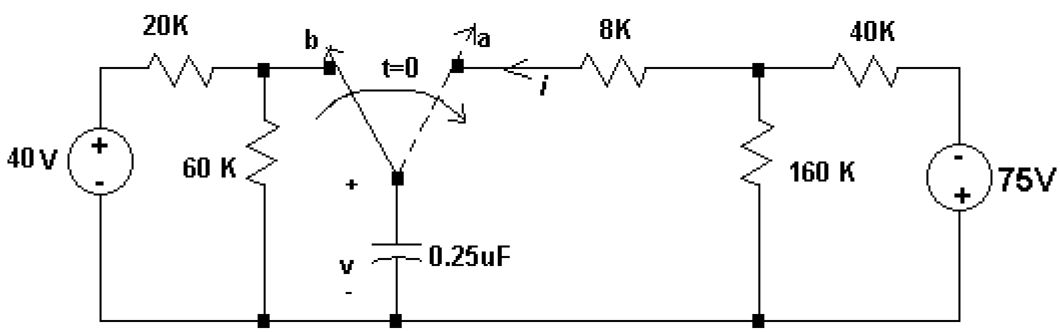
(c) Linear first order differential equation: $C\frac{dv}{dt}+\frac{v}{R}=I_s \rightarrow \frac{dv}{dt}+\frac{v}{RC}=\frac{I_s}{C}$

(d) τ=RC and $A=\frac{I_s}{C}$; $v(\infty)=\tau\cdot A=I_s R$

(f) Solution for voltage: $v_c(t)=v_c(\infty)+[v_c(0)-v_c(\infty)]=I_s R+\{v_c(0)-I_s R\}\cdot e^{-\frac{t}{RC}}$, $\tau=RC$

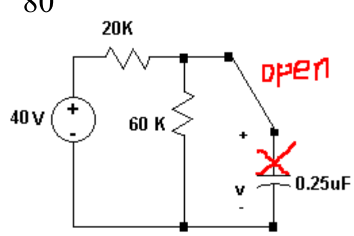
(g) Solution for current : $i_c(t)=C\frac{dv_c(t)}{dt}$

E.2. Example #1: The switch has been in position *b* for a long time. At t=0, it moves to position *a*. Find the voltage across the capacitor and the current through it for t>0.

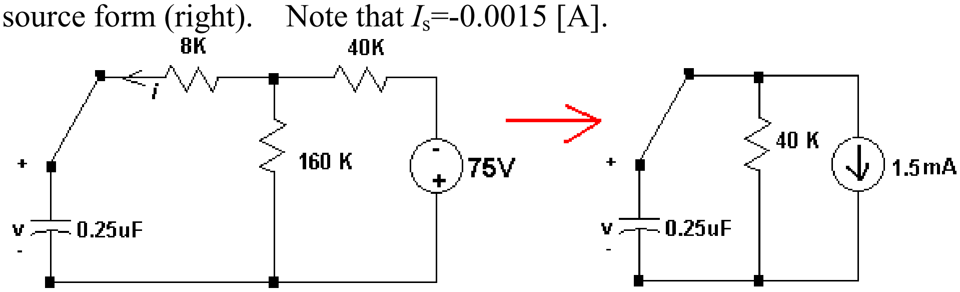


SOLUTION:

(a) for t<0: The capacitor is an open circuit in DC, therefore, the initial voltage across the capacitor is same as the voltage across the 60K resistor. Therefore, the initial capacitor voltage is, by voltage division: $v_c(0)=40\cdot\frac{60}{80}=30$ [V].



(b) for t>0: The circuit for t>0 (left) can be source transformed to have the usual RC-parallel with current source form (right). Note that I_s=-0.0015 [A].



(c) Node Voltage equation: $C\frac{dv}{dt}+\frac{v}{R}+I_s=0$ ---> $\frac{dv}{dt}+\frac{v}{RC}=-\frac{I_s}{C}$

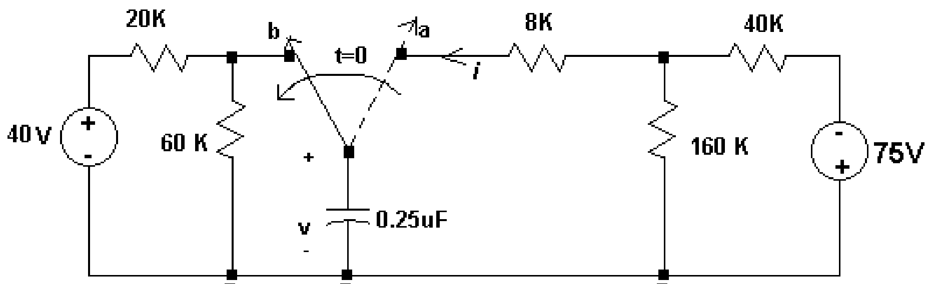
(d) τ=RC=0.01; $v(\infty)=\tau\cdot A=(0.01)\times(-6000)=-60$

(c) Voltage Solution: $v_c(t)=v_c(\infty)+[v_c(0)-v_c(\infty)]\cdot e^{-\frac{t}{RC}}=-60+[30-(-60)]e^{-100t}$

Therefore, $v_c(t)=-60+90\cdot e^{-100t}$

(d) Current: $i_c(t)=C\frac{dv}{dt}=-(0.25)\cdot10^{-6}\cdot9000\cdot e^{-100t}=-0.00225e^{-100t}$

E.3. Example #2: The switch has been in position *a* for a long time. At t=0, it moves to position *b*. Find the voltage across the capacitor and the current through it for t>0.



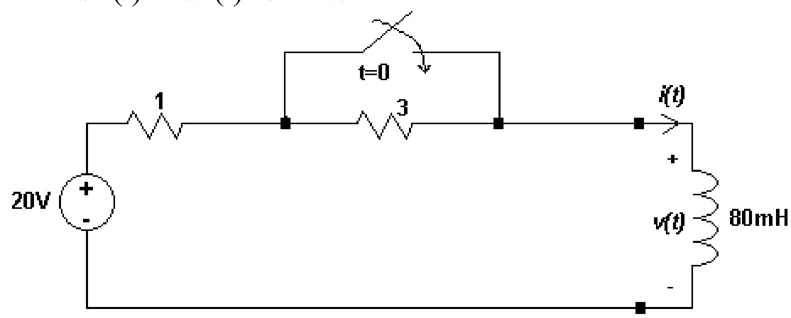
SOLUTION:

(a) for t<0:

(b) for t>0:

F. EXTRA PROBLEMS OF RL and RC RESPONSE

F.1. Example #1: The switch has been open for a long time. At t=0, the switch is closed. Find *v*(t) and *i*(t) for t>0.



SOLUTION: <u>(point: get i(t))</u>

(a) for t<0: The initial current in the inductor is 5[A]=$i(0^-)$

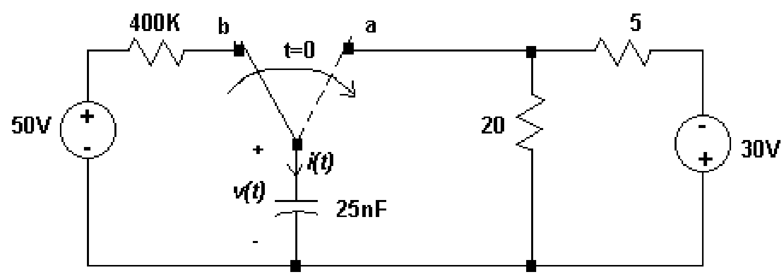
(b) for t>0: After switch close, the current becomes 20[A]=$i(\infty)$

Time constant $\tau = \frac{L}{R} = 80\times 10^{-3}$

Therefore, Current: $i(t) = 20 + (5-20)e^{-\frac{t}{0.08}} = 20 - 15e^{-12.5t}$

Voltage: $v(t) = L\frac{di}{dt} = (0.08)(-15\cdot -12.5)e^{-12.5t} = 15e^{-12.5t}$

F.2. Example #2: The switch has been in position *b* for a long time. At t=0, the switch is moved to position *a*. Find *v*(t) and *i*(t) for t>0.



SOLUTION: <u>(point: get v(t))</u>

(a) for t<0: The initial voltage across the capacitor is 50[V]=$v(0^-)$

(b) for t>0: After switch close, the voltage across the capacitor becomes

$$-30\cdot\frac{20}{25} = -24 = v(\infty)$$

Time constant $\tau = RC = 25\times 10^{-9}\cdot 4 = 10^{-7}$ (Note that: $R_{eq} = \frac{20\cdot 5}{20+5} = 4$)

Therefore, Voltage: $v(t) = -24 + (50-(-24))e^{-\frac{t}{10^{-7}}} = -24 + 74e^{-10^7 t}$

Current: $i(t) = C\frac{dv}{dt} = 25\times 10^{-9}\cdot 74\cdot(-10^7)\cdot e^{-10^7 t} = -18.5e^{-10^7 t}$

Class Note 28: Second-Order Natural Response

A. Review

A.1. A general second-order differential equation: $\frac{d^2x(t)}{dt^2} + 2\alpha\frac{dx(t)}{dt} + w_o{}^2 x(t) = K$

A.2. Then the solution form: (where $x(\infty) = K/w_o{}^2$)

For overdampled case: $x(t) = x(\infty) + A_1 e^{s_1 t} + A_2 e^{s_2 t}$

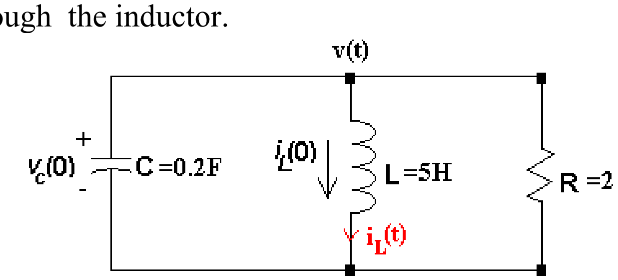
For underdamped case: $x(t) = x(\infty) + B_1 e^{-\alpha t}\cos\omega_d t + B_2 e^{-\alpha t}\sin\omega_d t$

For critically damped case: $x(t) = x(\infty) + D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$

A.3. We analyze four categories of first-order circuits. They are
(a)Paralle RLC Natural Response
(b)Series RLC Natural Response

B. Parallel RLC Natural Response

B.1. Example #1: Consider the parallel RLC circuit shown below. Let's assume that the initial conditions on the storage elements are: $i_L(0)$= -1[A] and $v_c(0)$=4 [V]. Find the node voltage v(t) and the current through the inductor.



SOLUTION:

(a) Node Voltage Equation:

$$\frac{v}{R} + \frac{1}{L}\int v dx + C\frac{dv}{dt} = 0 \rightarrow \frac{v}{2} + \frac{1}{5}\int v dx + 0.2\frac{dv}{dt} = 0$$

(b) Derivation with respect to time *t* :

$$0.2\frac{d^2v}{dt^2} + \frac{1}{2}\frac{dv}{dt} + \frac{v}{5} = 0 \rightarrow \frac{d^2v}{dt^2} + 2.5\frac{dv}{dt} + v = 0$$ (with α=1.25 and w_0^2=1)

(c) Characteristic equation:

$$s^2 + 2\alpha s + w_o{}^2 = 0 \rightarrow s^2 + 2.5s + 1 = 0$$

(d) Roots calculation: (s+2)(s+0.5)=0 ----> s_1=-2 and s_2= -0.5

[* or by $s_1 = -\alpha + \sqrt{\alpha^2 - w_o{}^2}$ and $s_2 = -\alpha - \sqrt{\alpha^2 - w_o{}^2}$]

(d) Damping condition check: $\alpha^2 > w_0^2$ (this is a overdamped case)

(e) Solution form: $v(t) = v(\infty) + A_1 e^{s_1 t} + A_2 e^{s_2 t} = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

(f) Constraints calculation:

$$v(0) = A_1 + A_2 = v_c(0) = 4 \quad \text{---(1)}$$

$\frac{dv(t)}{dt}\Big|_{t=0} = s_1 A_1 + s_2 A_2 \rightarrow$ to apply this we have a problem: the initial condition we have (other than the capacitor voltage) is **not** $\frac{dv(t)}{dt}\Big|_{t=0}$, **but** $i_L(0)$.

(g) Therefore, we slight change the original equation so that it includes the initial inductance current. So, let's change the original equation:

$$\frac{v}{R} + \frac{1}{L}\int v dx + C\frac{dv}{dt} = 0 \rightarrow \frac{v}{R} + i_L(t) + C\frac{dv}{dt} = 0$$

(h) Then, $\frac{v}{R} + i_L(t) + C\frac{dv}{dt} = 0 \rightarrow \frac{dv}{dt} = -\frac{1}{RC}v(t) - \frac{1}{C}[i_L(t)]$

(i) Now, we can calculate the constraints:

$$\frac{dv(t)}{dt}\Big|_{t=0} = -2A_1 - 0.5A_2 = -\frac{1}{RC}v(0) - \frac{1}{C}[i_L(0)] = -2.5(4) - 5(-1) = -5 \text{ ---(2)}$$

(j) From $A_1 + A_2 = 4$ and $2A_1 + 0.5A_2 = 5$, ----->A_1=2 and A_2=2

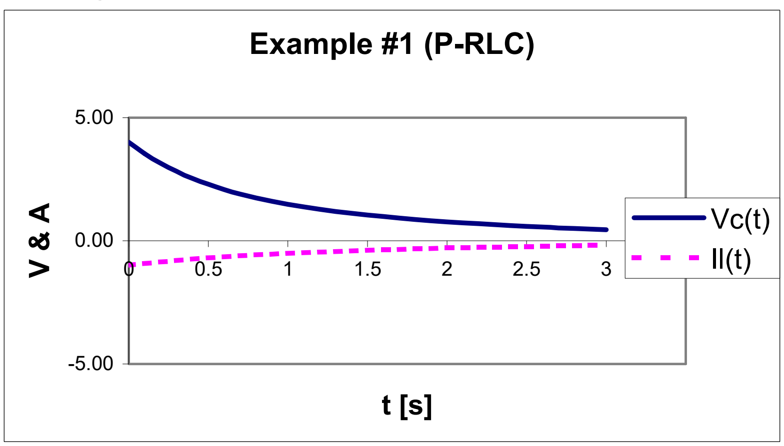
(k) Finally, from $v(t) = v(\infty) + A_1 e^{s_1 t} + A_2 e^{s_2 t} = 2e^{-2t} + 2e^{-0.5t}$

(l) For the inductor current: the inductor current is related to v(t) by $i_L(t) = \frac{1}{L}\int v(t)dt$.

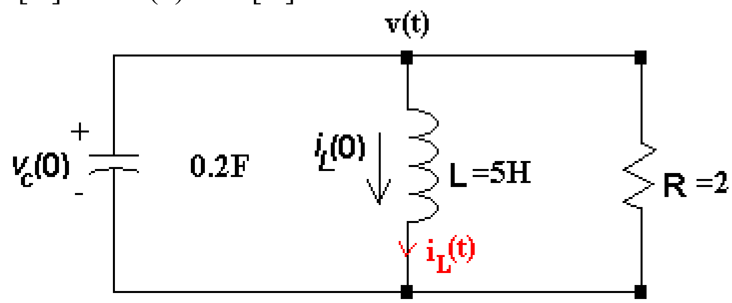
(m) Substitution of the voltage equation yields:

$$i_L(t) = \frac{1}{L}\int v(t)dt = \frac{1}{5}\int [2e^{-2t} + 2e^{-0.5t}]dt = -0.2e^{-2t} - 0.8e^{-0.5t}$$

(n) Sketch of the voltage and the inductor current



B.2. Example #2: Find *i*(t) **directly** (without getting v(t) first)
when $i_L(0)$= 0.3[A] and $v_c(0)$=1.2 [V]



SOLUTION:

(a) differential equation: $\frac{v}{R} + i_L(t) + C\frac{dv}{dt} = 0$

(b) Since $v = L\frac{di_L}{dt}$, then equation becomes:

$$\frac{L}{R}\frac{di_L}{dt} + i_L + LC\frac{d^2 i_L}{dt2} = 0 \rightarrow \frac{d^2 i_L}{dt2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{i_L}{LC} = 0$$

(c)

From initial voltage, $L\frac{di_L(0)}{dt} = 1.2$. Therefore, $\frac{di_L(0)}{dt} = \frac{1.2}{5} = 0.24$ with $i_L(0) = 0.3$

(b) Neper and Resonant Frequencies

$$\alpha = \frac{1}{2RC} = 1.25 \text{ and } \omega_0^{\ 2} = \frac{1}{LC} = 1$$

(c) Damping Types.

Since $\alpha^2 > \omega_0^{\ 2}$, it is overdamped case

(d) Solution form: $i(t) = i(\infty) + A_1 e^{-2t} + A_2 e^{-0.5t}$

(f) Constraints calculation:

$i(0) = A_1 + A_2 = 0.3$ ---(1)

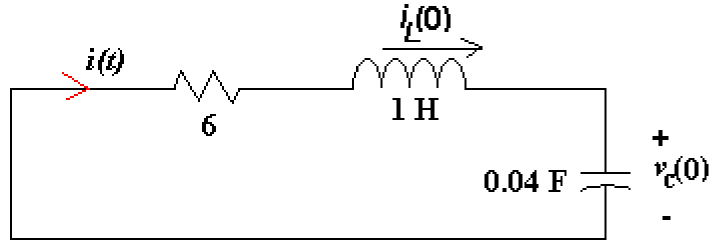
$\frac{di(t)}{dt}\Big|_{t=0} = -2A_1 - 0.5A_2 = 0.24$

Therefore, A_1=-0.26 and A_2=0.56

(g) Final answer: $\underline{i(t) = -0.26e^{-2t} + 0.56e^{-0.5t}}$

C. Series RLC Natural Response

C.1. Example #1: Consider the series RLC circuit shown below with the following parameter: $i_L(0)$= 4 [A] and $v_c(0)$= - 4 [V]. Find the current and the capacitor voltage.



SOLUTION:

(a) KVL around the loop:

$$Ri + L\frac{di}{dt} + v_c(t) = 0 \quad \text{or} \quad Ri + L\frac{di}{dt} + \frac{1}{C}\int i(t)dt = 0$$

(b) Derivation with respect to time t :

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{i}{LC} = 0 \rightarrow \frac{d^2i}{dt^2} + 6\frac{di}{dt} + 25i = 0 \quad \text{(with } \alpha\text{=3 and } w_0^2\text{=25)}$$

(c) Characteristic equation:

$$s^2 + 2\alpha s + w_o^{\ 2} = 0 \rightarrow s^2 + 6s + 25 = 0$$

(d) Roots calculation: By $s_1 = -\alpha + \sqrt{\alpha^2 - w_o^{\ 2}}$ and $s_2 = -\alpha - \sqrt{\alpha^2 - w_o^{\ 2}}$

s_1=-3+j4 and s_2=-3-j4

(d) Damping condition check: $\alpha^2 < w_0^2$ (this is a underdamped case)

(e) Solution form:

$$i(t) = i(\infty) + B_1 e^{-\alpha t}\cos\omega_d t + B_2 e^{-\alpha t}\sin\omega_d t, \text{ where } w_d = \sqrt{25-9} = 4$$

So, $i(t) = B_1 e^{-3t}\cos 4t + B_2 e^{-3t}\sin 4t$

(f) Constraints calculation:

$$i(0) = x(\infty) + B_1 = B_1 = 4$$

$$\frac{di(t)}{dt}\Big|_{t=0} = -\alpha\ B_1 + \omega_d B_2 = -3B_1 + 4B_2 = -12 + 4B_2$$

(g) From the original equation: $Ri + L\dfrac{di}{dt} + v_c(t) = 0$

(h) $Ri + L\dfrac{di}{dt} + v_c(t) = 0 \rightarrow \dfrac{di}{dt} = -\dfrac{R}{L}i - \dfrac{v_c(t)}{L}$

(i) Now, we can calculate the constraints:

$$\frac{di(t)}{dt}\Big|_{t=0} = -12 + 4B_2 = -\frac{6}{1}i(0) - \frac{v(0)}{1} = -\frac{24}{1} + 4 = -20$$

(j) Therefore, B_2=-2.

(k) Finally, $i(t) = 4e^{-3t}\cos 4t - 2e^{-3t}\sin 4t$

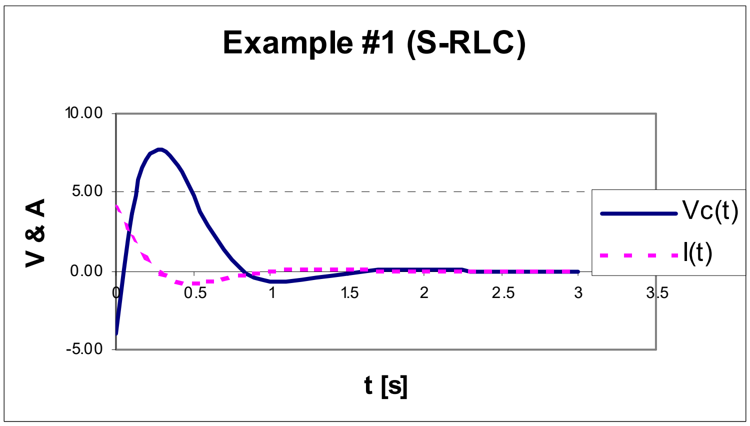
(l) For the voltage across the capacitor: From $Ri + L\dfrac{di}{dt} + v_c(t) = 0$,

$$v_c(t) = -Ri - L\frac{di}{dt} = -6(4e^{-3t}\cos 4t - 2e^{-3t}\sin 4t) - \frac{di}{dt}$$

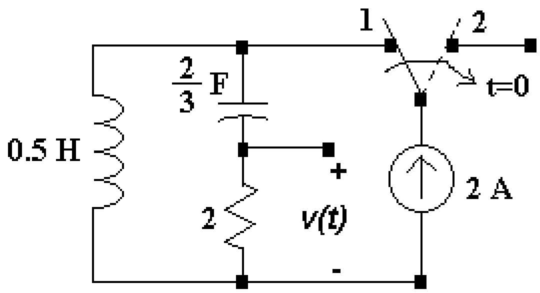
(m) Simplification yields:

$$v_c(t) = -4e^{-3t}\cos 4t + 22e^{-3t}\sin 4t$$

(n) Sketch of the voltage and the inductor current

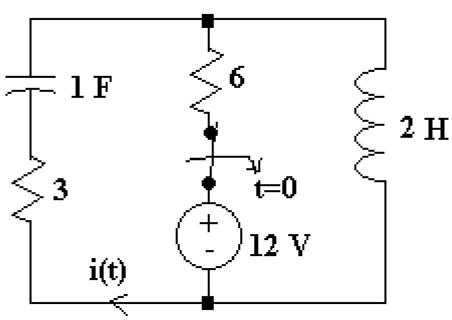


C.2 Example #2: The switch in the circuit has been in position 1 for a long time. At t=0, the switch moves from position 1 to 2. Find *v*(t).



SOLUTION:

C.3. Example #3: The switch in the circuit has been closed for a long time. At t=0, the switch opens. Find *i*(t).



SOLUTION:

Class Note 29: **Step Responses of Parallel and Series RLC circuits**

A. Review

A.1. A general second-order differential equation: $\frac{d^2x(t)}{dt^2} + 2\alpha \frac{dx(t)}{dt} + w_o^{\ 2} x(t) = K$

A.2. Then the solution form: (where $x(\infty) = K / w_o^{\ 2}$)

For overdampled case: $x(t) = x(\infty) + A_1 e^{s_1 t} + A_2 e^{s_2 t}$

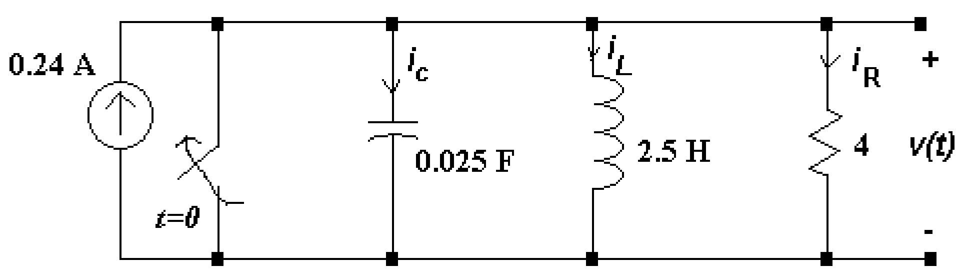
For underdamped case: $x(t) = x(\infty) + B_1 e^{-\alpha t} \cos \omega_d t + B_2 e^{-\alpha t} \sin \omega_d t$

For critically damped case: $x(t) = x(\infty) + D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$

A.3. We analyze four categories of first-order circuits. They are
(a)Paralle RLC Natural Response
(b)Series RLC Natural Response

B. Step Response of Parallel RLC

B.1 Example #1: Find $i_L(t)$.



(a) t<0:
i. There is no initial current in the inductor, therefore $i_L(0)$=0.
ii. The initial voltage across the capacitor is zero, therefore, v(0)=0.

REGULAR APPROACH [Get v(t) first using node voltage equation, then get i(t)]

(b) Node voltage equation:

$$-0.24 + \frac{v}{R} + \frac{1}{L}\int v dx + C\frac{dv}{dt} = 0$$

Differentiation with respect to *t*: $\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0$

(b) Neper and Resonant Frequencies

$$\alpha = \frac{1}{2RC} = \frac{1}{2\cdot 4\cdot(0.025)} = 5 \text{ and } \omega_0^{\,2} = \frac{1}{LC} = \frac{1}{(2.5)(0.025)} = 16$$

(c) Final value: $v(\infty) = K/w_o^{\,2} = 0$

(d) The roots of the characteristic equation:

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^{\,2}} = -5 + 3 = -2 \quad \text{and} \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^{\,2}} = -5 - 3 = -8$$

(e) Damping Types.

Since $\alpha^2 > \omega_0^{\,2}$, it is over damping response

(f) Solution form: $v(t) = v(\infty) + A_1 e^{s_1 t} + A_2 e^{s_2 t}$

(g) Since $v(\infty) = \dfrac{K}{w_o^{\,2}} = 0$, $\underline{v(t) = A_1 e^{-2t} + A_2 e^{-8t}}$

(h) Let's apply the two constraints for A_1 and A_2.

From $v(0) = v(\infty) + A_1 + A_2$. $v(0) = A_1 + A_2 = 0$ ----(1)

For $\dfrac{dv(t)}{dt}\Big|_{t=0} = s_1 A_1 + s_2 A_2$ constraint, we slightly change our original equation:

$$-0.24 + \frac{v}{R} + \frac{1}{L}\int v\,dx + C\frac{dv}{dt} = 0 \rightarrow -0.24 + \frac{v}{R} + i_L + C\frac{dv}{dt} = 0$$

Then, $\dfrac{dv}{dt} = -\dfrac{v}{RC} - \dfrac{i_L}{C} + \dfrac{0.24}{C}$.

Therefore, $\dfrac{dv}{dt}\Big|_{t=0} = -\dfrac{v(0)}{RC} - \dfrac{i_L(0)}{C} + \dfrac{0.24}{C} = \dfrac{0.24}{C} = \dfrac{0.24}{0.025} = 9.6$

So the relationship goes like this:

$$\frac{dv(t)}{dt}\Big|_{t=0} = s_1 A_1 + s_2 A_2 = -2A_1 - 8A_2 = 9.6 \text{ ------(2)}$$

From (1) and (2): we have: A_1=1.6 and A_2=-1.6

(i) Therefore, voltage: $v(t) = 1.6e^{-2t} - 1.6e^{-8t}$ [V]

(j) Finally, the current: From $-0.24 + \dfrac{v}{R} + i_L + C\dfrac{dv}{dt} = 0$

$$i_L = 0.24 - \frac{v}{R} - C\frac{dv}{dt} = 0.24 - (0.25)(1.6e^{-2t} - 1.6e^{-8t}) - (0.025)(-3.2e^{-2t} + 12.8e^{-8t})$$

$$\underline{= 0.24 - 0.48e^{-2t} + 0.08e^{-8t}}$$

Alternatively, you can do this way:

$$i(t) = \frac{1}{L}\int_0^t v(\tau)d\tau = 0.4\int_0^t [e^{-2\tau} - e^{-8\tau}]d\tau = 0.24 - 0.48e^{-2t} + 0.08e^{-8t}$$

*NOTE: See an alternative approach next page.

ALTERNATIVE APPROACH (DIRECT METHOD)

(b) t>0: Since our target is $i_L(t)$, let's express every term in terms of the current.

KCL Equation: $-0.24 + C\frac{dv(t)}{dt} + i_L(t) + \frac{v(t)}{R} = 0$

Since $v(t) = L\frac{di_L}{dt}$, $C\frac{dv}{dt} = LC\frac{d^2 i_L}{dt^2}$

Therefore ,the KCL equation becomes:

$$LC\frac{d^2 i_L}{dt^2} + \frac{L}{R}\frac{di_L}{dt} + i_L = 0.24 \rightarrow \frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{1}{LC}i_L = \frac{0.24}{LC} = 3.84$$

(b) Neper and Resonant Frequencies

$$\alpha = \frac{1}{2RC} = 5 \text{ and } \omega_0^2 = \frac{1}{LC} = 16$$

(c) Final value: $i(\infty) = K/w_o^2 = 0.24$

(d) The roots of the characteristic equation:

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -5 + 3 = -2 \quad \text{and } s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -5 - 3 = -8$$

(e) Damping Types.

Since $\alpha^2 > \omega_0^2$, it is over damping response

(f) Solution form: $v(t) = v(\infty) + A_1 e^{s_1 t} + A_2 e^{s_2 t}$

(g) Since $v(\infty) = \frac{K}{w_o^2} = 0.24$, $v(t) = 0.24 + A_1 e^{-2t} + A_2 e^{-8t}$

(h) Let's apply the two constraints for A_1 and A_2.

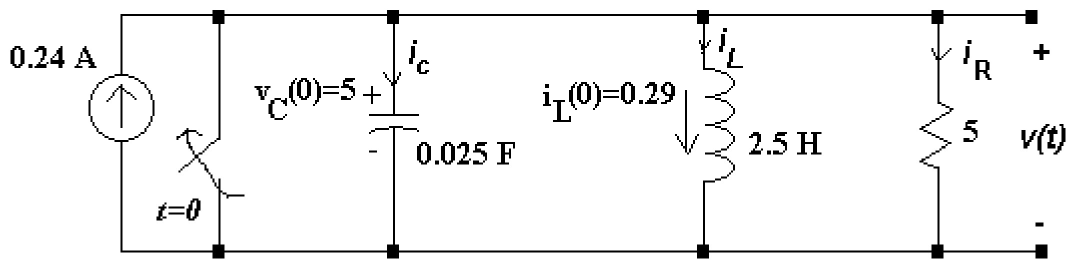
From $i(0) = i(\infty) + A_1 + A_2$. $i(0) = 0.24 + A_1 + A_2 = 0$----(1)

From $\frac{di(t)}{dt}\Big|_{t=0} = \frac{v(t)}{L}\Big|_{t=0} = 0 = s_1 A_1 + s_2 A_2 = -2A_1 - 8A_2$.-------(2)

From (1) and (2): we have: A_1=-0.48and A_2=0.08

(i) Finally: $i(t) = 0.24 - 0.48e^{-2t} + 0.08e^{-8t}$ [A]

B.2. Example #2: Energy is stored in the circuit before the DC current source is applied, with $i_L(0) = 0.29$ [A] and $v_C(0) = 5$ [V]. Find $i_L(t)$.



SOLUTION (Use **Direct** approach)

$$LC\frac{d^2 i_L}{dt^2} + \frac{L}{R}\frac{di_L}{dt} + i_L = 0.24 \rightarrow \frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{1}{LC} i_L = \frac{0.24}{LC} = 3.84$$

From initial voltage, $L\frac{di_L(0)}{dt} = 5$. Therefore, $\frac{di_L(0)}{dt} = \frac{5}{2.5} = 2$ with $i_L(0) = 0.29$

(b) Neper and Resonant Frequencies

$$\alpha = \frac{1}{2RC} = 4 \text{ and } \omega_0^2 = \frac{1}{LC} = 16$$

(c) Damping Types.

Since $\alpha^2 = \omega_0^2$, it is critically damping response

(d) Solution form

From $z(t) = x(\infty) + x(t)$ and , $x(t) = D_1 te^{-\alpha t} + D_2 e^{-\alpha t}$

$i_L(t) = i_L(\infty) + D_1 te^{-4t} + D_2 e^{-4t}$

Therefore, $i_L(t) = 0.24 + D_1 te^{-4t} + D_2 e^{-4t}$

(e) Let's apply the two constraints for A_1 and A_2.

From $x(0) = x(\infty) + D_2$, $i_L(0) = 0.24 + D_2 = 0.29$

From $\frac{dx(0)}{dt} == D_1 - \alpha D_2$, $\frac{di_L(0)}{dt} = -D_1 + 4D_2 = 2$

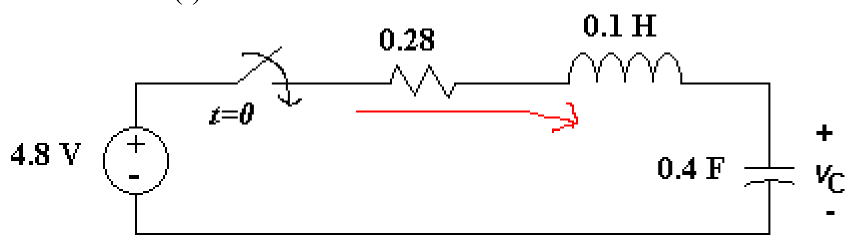
From these two constraints, we have:

$D_1 = 0.05$ and $D_2 = 0.51$

(f) Final answer: $i_L(t) = 0.24 + 0.05te^{-4t} + 0.51e^{-4t}$ [A]

C. Step Response of Series RLC

C.1. Example #1: Find $v_C(t)$.



(a) Differential equation:

$$-48 + Ri(t) + L\frac{di}{dt} + v_c(t) = 0$$

(b) Since the question is to find the voltage, let's use the direct method. Instead of getting current and then voltage.

The current through the capacitor is: $i = C\frac{dv_c}{dt}$

Therefore, the differential equation becomes: $\frac{d^2v_c}{dt^2} + \frac{R}{L}\frac{dv_c}{dt} + \frac{1}{LC}v_c = \frac{4.8}{LC} = 120$

(c) Neper and Resonant Frequencies

$\alpha = \frac{R}{2L} = \frac{0.28}{2\cdot(0.1)} = 1.4$ ---->$\alpha^2 = 1.96$ and $\omega_0^{\ 2} = \frac{1}{LC} = \frac{1}{(0.1)\cdot(0.4)} = 25$

And $v(\infty) = 4.8$

(d) The roots of the characteristic equation:

$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^{\ 2}} = -1.4 + j4.8$ and $s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^{\ 2}} = -1.4 - j4.8$

(e) Damping Types.

Since $\alpha^2 < \omega_0^{\ 2}$, it is under damping response

constraint 3: $\omega_d = \sqrt{w_o^{\ 2} - \alpha^2} = 4.8$

(f) Solution form

From $x(t) = x(\infty) + x_c(t)$ ----> $v(t) = 4.8 + B_1e^{-1.4t}\cos 4.8t + B_2e^{-1.4t}\sin 4.8t$

(g) Let's apply the two constraints for B_1 and B_2.

From $v(0) = v(\infty) + B_1$, B_1=-4.8

For $\frac{dv(0)}{dt} = -\alpha_1 B_1 + \omega_d B_2$,

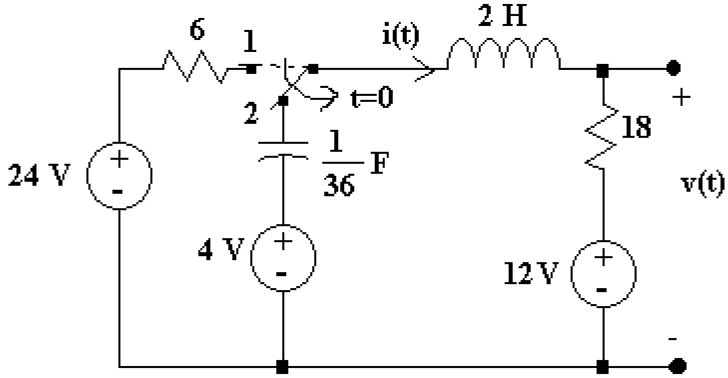
from , $i = C\frac{dv_c}{dt}$ --> $\frac{dv}{dt}\Big|_{t=0} = \frac{i(0)}{C} = 0$. Therefore, $-1.4B_1 + 4.8B_2 = 0$

From these two constraints, we have: B_1= -4.8 and B_2= -1.4

(h) Then, current: $\underline{v_c(t) = 4.8 - 4.8e^{-1.4t}\cos 4.8t - 1.4e^{-1.4t}\sin 4.8t}$ [V]

Class Note 30: RLC Response Extra Problems

1. The switch in the circuit has been in position 1 for a long time. At t=0, it moves from position 1 to position 2. Compute *i*(t) for t>0 and use this current to determine the voltage *v*(t).



2. The switch in the circuit has been in position 1 for a long time. At t=0, it moves from position 1 to position 2. Compute *v*(t) for t>0.