S-domain Analysis

1. Laplace Transformation –summary

1. Integral Transformation:

Laplace transformation belongs to a class of analysis methods called integral transformation which are studied in the field of operational calculus. These methods include the Fourier transform, the Mellin transform, etc. In each method, the idea is to transform a difficult problem into an easy problem. For example, taking the Laplace transform of both sides of a linear, ordinary differential equation results in an algebraic problem. Solving algebraic equations is usually easier than solving differential equations. The one-sided Laplace transform defined in 3 below, is valid over the interval [0, ∞). This means that the domain of integration includes its left end point. This is why most authors use the term (0-) to represent the bottom limit of the Laplace integral.

2. Comparison of a few Transform methods:

A. FOURIER TRANSFORM: $X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$

(1) Usually for energy signals, in the limit for singularity functions, periodic signals, causal or non-causal signals.
(2) Steady state circuit analysis, algebraic differential solutions.
(3) It can be used to perform convolution very fast for discrete signals.
(4) Fourier transform is a function of one variable, w, and plots of it have a lot of meaning.

B. LAPLACE TRANSFORM: $X(s) = \int_{0-}^{\infty} x(t) e^{-st} dt$

(1) Usually for signals starting at time zero, exists for many non-energy, non-power signals for which the Fourier transform does not exists.
(2) Transient and steady state analysis; initial conditions in differential equations handled algebraic differential equations.
(3) Analysis tool, but not useful for fast convolution.
(4) Laplace transform is function of one complex variable, g, much harder to plot and the plots have much less usefulness.

C. Z-TRANSFORMATION : $Y(z) = \sum_{k=0}^{\infty} y(k) z^{-k}$

(1) The method of z-transformation does for discrete systems what Laplace transformation does for continuous systems.

3. Definition:

$$L(f(t)) = \int_{0-}^{\infty} f(t) e^{-st} dt = F(s)$$

$t=0 \rightarrow e^{-st}=0$

4. Laplace transformation of selected function:

$\mathcal{L}(\delta(t)) = \int_{0-}^{\infty} \delta(t) e^{-st} dt = \int_{0-}^{\infty} \delta(t) dt = 1$

a. $L\{\delta(t)\} = 1$

$$\int_{0^-}^{\infty} u(t) e^{-st} dt = \int_{0^-}^{\infty} 1 \cdot e^{-st} dt = \left. \frac{e^{-st}}{-s} \right]_{0^-}^{\infty} = \frac{1}{s}$$

b. $L\{u(t)\} = \frac{1}{s}$

$$\int_{0^-}^{\infty} e^{-at} e^{-st} dt = \int_{0^-}^{\infty} e^{-(a+s)t} dt = \frac{1}{s+a} \left. e^{-(s+a)t} \right|_{0}^{\infty}$$

$$= \frac{1}{s+a}$$

c. $L\{e^{-at}\} = \frac{1}{s+a}$

$$e^{j\omega t} = \cos \omega t + j \sin \omega t$$

$$\int_{0^-}^{\infty} e^{j\omega t} e^{-st} dt = \int_{0^-}^{\infty} e^{-(s-j\omega)t} dt$$

$$= \frac{1}{s-j\omega} \rightarrow \underbrace{\frac{s}{s^2+\omega^2}}_{\mathcal{L}(\cos \omega t)} + j \underbrace{\frac{\omega}{s^2+\omega^2}}_{\mathcal{L}[\sin \omega t]}$$

d. $L\{\cos wt\} = \frac{s}{s^2 + w^2}$

e. $L\{\sin wt\} = \frac{w}{s^2 + w^2}$

$$\int_{0^-}^{\infty} t e^{-st} dt = t \left(-\frac{e^{-st}}{s}\right)' dt$$

$$= \left. t \frac{e^{-st}}{-s} \right]_{0^-}^{\infty} - \int -\frac{e^{-st}}{s} dt$$

$$= \left. \frac{e^{-st}}{s^2} \right]_{0}^{\infty} = \frac{1}{s^2}$$

f. $L\{t\} = \frac{1}{s^2}$

g. $L\{t^2\} = \frac{2}{s^3}$ ------> *general form: $L\{t^n\} = \frac{n!}{s^{n+1}}$

5. Properties of Laplace Transformations:

a. Linearity: $L\{kf(t)\} = kF(s)$

b. Scaling: $L\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$

c. Time Shift: $L\{f(t-a)u(t-a)\} = e^{-as}F(s)$

*also, $L\{u(t-a)\} = \frac{e^{-as}}{s}$

d. Frequency Shift: $L\{e^{-at}f(t)\} = F(s+a)$

e. Time Differentiation: $L\{f'(t)\} = sF(s) - f(0^-)$

*general form: $\frac{d^n f}{dt^n} = s^n F(s) - s^{n-1}f(0^-) - s^{n-2}f'(0^-) - \ldots - f^{(n)}(0^-)$

f. Time Integration: $L\{\int_0^t f(x)dx\} = \frac{F(s)}{s}$

g. Frequency Differentiation: $L\{t \cdot f(t)\} = -F'(s)$

$\int_0^\infty t F(t) e^{st} dt$

$= t F(t)\Big]_0^\infty - \int_0^\infty F(t) dt$

$= 0 - F(s)$

h. Frequency Integration: $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)du$

i. Time Periodicity: $L\{f(t) = f(t+nT)\} = \frac{F_1(s)}{1-e^{-Ts}}$

j. Initial Value: $f(0^+) = \lim_{s\to\infty} sF(s)$

k. Final Value: $f(\infty) = \lim_{s\to 0} sF(s)$

l. Convolution: $L\{f(t) * g(t)\} = F(s)G(s)$

$\mathcal{L}\{f'(t)\} = \int_{0^-}^{\infty} f'(t) e^{-st} dt = f(t) e^{-st}\Big]_0^\infty - \int_0^\infty f(t)(-s e^{-st}) dt$

u', v

$= -f(0) + s\int_0^\infty f(t) e^{-st} dt$

$= -f(0) + sF(s)$

2. Inverse Laplace Transformation

1. From the definition of Laplace transform, $L\{f(t)\} = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$, the inverse Laplace transform is given by,

$$L^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi j}\int_{\sigma 1 - j\infty}^{\sigma 1 + j\infty} F(s)e^{st}ds$$

where the integration is performed along a straight line (σ1+jw, for -∞ <w<∞) in the region of convergence, $\sigma 1 < \sigma_c$. This involves some knowledge about complex analysis beyond the scope of this course. For this reason, we use the properties of the Laplace transformation (see note 14) for inverse transformation. **In other words, we try to match function F(s) to an entry of LT.**

2. Finding the inverse Laplace transform of F(s) involves two steps.
   (a) Decompose F(s) into simple terms using partial fraction expansion
   (b) Find the inverse of each term by matching the entries.
   *NOTE: Software packages such as Matlab, Mathcad, and Maple are capable of finding partial fraction expansions quite easily. ω

$\frac{3 \cdot 2}{s^2+2^2}$

3. Simple Example

   Find the inverse Laplace transform of $F(s) = \frac{3}{s} - \frac{5}{s+1} + \frac{6}{s^2+4}$

   $B \quad \omega$

   Solution: $f(t) = L^{-1}\{F(s)\} = L^{-1}\{\frac{3}{s}\} - L^{-1}\{\frac{5}{s+1}\} + L^{-1}\{\frac{6}{s^2+4}\} = 3u(t) - 5e^{-t} + 3\sin 2t$, t>0

4. Simple Root Example

   Find f(t) given that $F(s) = \frac{s^2+12}{s(s+2)(s+3)}$

   Solution:

   Step 1: Partial fraction expansion: $F(s) = \frac{s^2+12}{s(s+2)(s+3)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3}$

   Step 2: There are two approaches available.

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   Residue Method:

$$A = sF(s)|_{s=0} = \frac{s^2+12}{(s+2)(s+3)}|_{s=0} = 2$$

$$B = (s+2)F(s)|_{s=-2} = \frac{s^2+12}{s(s+3)}|_{s=-2} = -8$$

$$C = (s+3)F(s)|_{s=-3} = \frac{s^2+12}{s(s+2)}|_{s=-3} = 7$$

   Algebraic Method: Multiplying both sides by $s(s+2)(s+3)$ gives

$$s^2 + 12 = A(s+2)(s+3) + Bs(s+3) + Cs(s+2) = (A+B+C)s^2 + (5A+3B+2C)s + 6A$$

Therefore, A+B+C=1, 5A+3B+2C=0, and 6A=12.
Finally, A=2, B=-8, and C=7

Step 3: From $F(s)=\frac{2}{s}-\frac{8}{s+2}+\frac{7}{s+3}$, $f(t)=2u(t)-8e^{-2t}+7e^{-3t}$, t>0

$\frac{As+B}{(s+2)^2}=\frac{A(s+2)+B-2A}{(s+2)^2}=\frac{A}{s+2}+\frac{B-2A}{(s+2)^2}$

5. Repeated Root Example

Calculate v(t) given that $V(s)=\frac{10s^2+4}{s(s+1)(s+2)^2}$

Solution:

Step 1: $V(s)=\frac{10s^2+4}{s(s+1)(s+2)^2}=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{(s+2)^2}+\frac{D}{s+2}$

Step 2:

Skip

Residue Method: $A=sV(s)|_{s=0}=\frac{10s^2+4}{(s+1)(s+2)^2}|_{s=0}=1$

$B=(s+1)V(s)|_{s=-1}=\frac{10s^2+4}{s(s+2)^2}|_{s=-1}=-14$

$C=(s+2)^2V(s)|_{s=-2}=\frac{10s^2+4}{s(s+1)}|_{s=-2}=22$

NOTE HERE------> $D=\frac{d}{ds}[(s+2)^2V(s)]|_{s=-2}=\frac{d}{ds}[\frac{10s^2+4}{s^2+s}]|_{s=-2}=13$

multiply (s+2)^2, derivative, then assess @s=-2

Algebraic Method: Multiplying both sides by $s(s+1)(s+2)^2$ gives

$10s^2+4=(A+B+D)s^3+(5A+4B+C+3D)s^2+(8A+4B+C+2D)s+4A$

Solving 4=4A, 0=8A+4B+C+2D, 10=5A+4B+C+3D, and 0=A+B+D gives

A=1, B=~~-1~~, C=22, and D=13.

B=-14

Step 3: From $V(s)=\frac{10s^2+4}{s(s+1)(s+2)^2}=\frac{1}{s}-\frac{14}{s+1}+\frac{22}{(s+2)^2}+\frac{13}{s+2}$,

$f(t)=u(t)-14e^{-t}+13e^{-2t}+22te^{-2t}$, t>0

6. Complex Root Example

(s+4)^2 + 9 --> (s+4)^2 - (j3)^2 =0
a^2 - b^2 = (a+b)(a-b)
s1=-4+j3
s2=-4-j3

Calculate i(t) given that $I(s)=\frac{20}{(s+3)(s^2+8s+25)}$.

Observation: I(s) has a pair of complex roots at $s^2+8s+25=0$ or $s=-4\pm j3$

Solution:

Step 1: $I(s)=\frac{20}{(s+3)(s^2+8s+25)}=\frac{A}{s+3}+\frac{Bs+C}{s^2+8s+25}$

Step 2:

skip

~~Residue + Algebraic Method~~: $A=(s+3)I(s)|_{s=-3}=\frac{20}{s^2+8s+25}|_{s=-3}=2$

Then, let's substitute two specific value of s for two simultaneous equations.

s=0: $I(0)=\frac{20}{(3)(25)}=\frac{2}{3}+\frac{C}{25}$, therefore C= - 10 skip

s=1: $I(1)=\dfrac{20}{(4)(1+8+25)}=\dfrac{2}{1+3}+\dfrac{B-10}{1+8+25}$, therefore B= - 2

Algebraic Method: Multiplying both sides by $(s+3)(s^2+8s+25)$ gives

$$20=(A+B)s^2+(8A+3B+C)s+25A+3C$$

Finally, A=2, B=-2, and C=-10

Step 3: From $I(s)=\dfrac{20}{(s+3)(s^2+8s+25)}=\dfrac{2}{s+3}+\dfrac{-2s-10}{s^2+8s+25}$, we change the equation so that it matches with the entries of the properties of transformation.

$$I(s)=\frac{2}{s+3}+\frac{-2s-10}{s^2+8s+25}=\frac{2}{s+3}+\frac{-2(s+4)-2}{(s+4)^2+9}=\frac{2}{s+3}-\frac{-2(s+4)}{(s+4)^2+9}+\frac{2}{3}\frac{3}{(s+4)^2+3^2}$$

Therefore, $i(t)=2e^{-3t}-2e^{-4t}\cos 3t+(2/3)e^{-4t}\sin 3t$. Applying the trigonometry formulae of $A\cos x+B\sin x=\sqrt{A^2+B^2}\cos[x-\tan^{-1}\left(\dfrac{B}{A}\right)]$ gives,

Just leave it here is OK

$$i(t)=2e^{-3t}-2.108e^{-4t}\cos(3t-18.43)$$

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7. Transformation Formula for Complex Root Case

From above, we derived $i(t)=2e^{-3t}-2.108e^{-4t}\cos(3t-18.43)$

from $I(s)=\dfrac{2}{s+3}+\dfrac{-2s-10}{s^2+8s+25}$

In other words, $L^{-1}\left(\dfrac{2s+10}{s^2+8s+25}\right)=2.108e^{-4t}\cos(3t-18.43)$

Let's try to form a useful formula from this observation.

(a) Let's expand the s-domain function.

$$\frac{2s+10}{s^2+8s+25}=\frac{2s+10}{(s+4)^2+9}=\frac{2s+10}{(s+4)^2+3^2}=\frac{2s+10}{(s+4)^2-(j3)^2}$$

a^2 - b^2

a-b

$$=\frac{A+jB}{(s+4-j3)}+\frac{A-jB}{(s+4+j3)}$$

a+b

(b) By algebraic approach we could get the relationships of:

8A

$2s+10=(A+jB)(s+4+j3)+(A-jB)(s+4-j3)=2As+\cancel{8}-6B$

--> A=1, B= -1/3

$K=|K|e^{j\theta}$

(c) The equation in (a) could be rewritten as:

$$\frac{2s+10}{s^2+8s+25}=\frac{A+jB}{(s+4-j3)}+\frac{A-jB}{(s+4+j3)}=\frac{K}{(s+\alpha-j\beta)}+\frac{K^*}{(s+\alpha+j\beta)},\ \text{where } K=1-j\frac{1}{3}$$

and $\alpha=4$, $\beta=3$

(d) The inverse transformation of (c) is:

$Ke^{-(\alpha-j\beta)t}+K^*e^{-(\alpha+j\beta)t}=|K|e^{j\theta}e^{-(\alpha-j\beta)t}+|K|e^{-j\theta}e^{-(\alpha+j\beta)t}$

where $K=|K|\angle\theta$ and $K^*=|K|\angle-\theta$ $\quad K^*=|K|e^{-j\theta}$

(e) Then, equation (d) becomes:

$|K|e^{j\theta}e^{-(\alpha-j\beta)t}+|K|e^{-j\theta}e^{-(\alpha+j\beta)t}=|K|e^{-\alpha t}\{e^{j(\beta t+\theta)}+e^{-j(\beta t+\theta)}\}=\underline{2|K|e^{-\alpha t}\cos(\beta t+\theta)}$

(f) If you substitute K, α, and β with their values,

we finally get the time function as:

$$2|K|e^{-\alpha t}\cos(\beta t+\theta)=2(\sqrt{1^2+(\frac{1}{3})^2}e^{-4t}\cos(3t-18.43)=2.108e^{-4t}\cos(3t-18.43)$$

(g) BOTTOM LINE: THE DISTINCT COMPLEX FORMULA

F(s)	$K=\vert K\vert\angle\theta$	f(t)
$\dfrac{K}{(s+\alpha-j\beta)}+\dfrac{K^*}{(s+\alpha+j\beta)}$		$\underline{2\vert K\vert e^{-\alpha t}\cos(\beta t+\theta)}$

3. s-Domain Analysis: Example Problems

A. s-domain circuit

Related operation transformations for inductor/capacitor:

$$L\{f'(t)\} = sF(s) - f(0^-) \text{ and } L\{\int_0^t f(x)dx\} = \frac{F(s)}{s}$$

(ex) From $v = L\frac{di}{dt}$, $V(s) = L\{sI(s) - i(0-)\} = sLI(s) - Li(0-)$

*note: Source transform

$i = C\frac{dv}{dt}$

$I(s) = C[sV(s) - V(0)]$

$= CsV(s) - CV(0)$

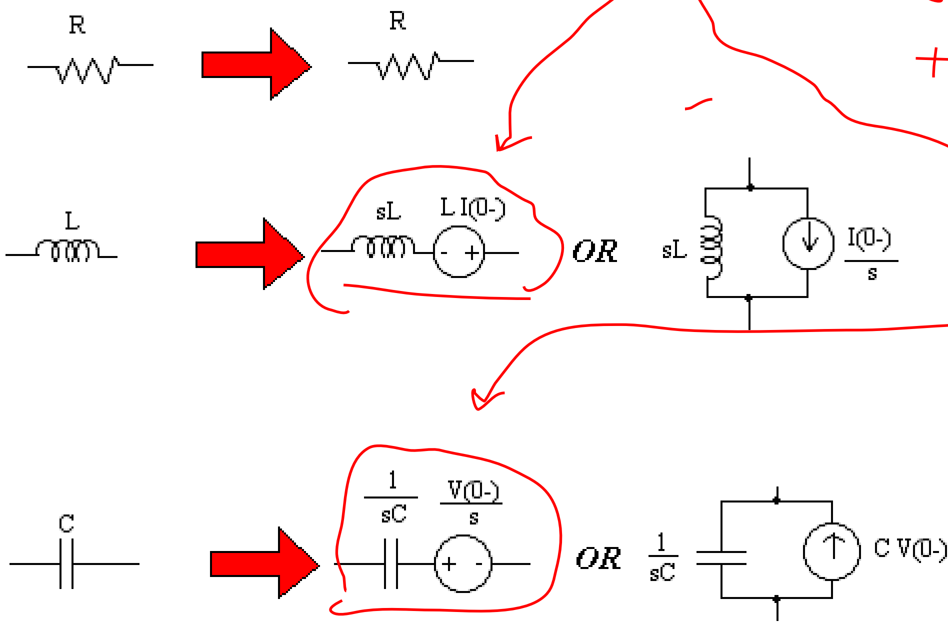
$\rightarrow V(s) = \frac{1}{Cs} I(s) + \frac{V(0)}{s}$

Time Domain **S – Domain**

Resistor:

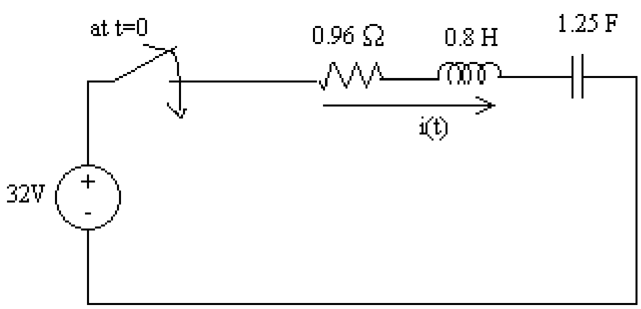
Inductor:

Capacitor:



B. EXAMPLE PROBLEMS:

1. The switch in the circuit closed at t=0. Find current i(t) at t>0 using s-domain analysis



32V from t = 0

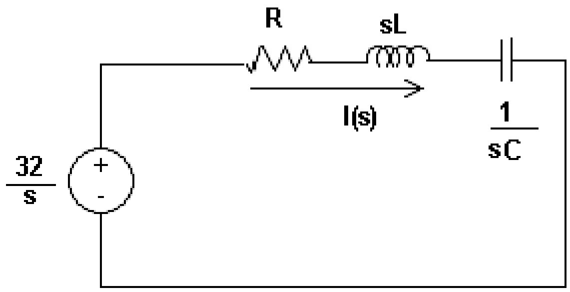
32u(t)

SOLUTION

$I_0 = 0$ $V_0 = 0$

t<0: ~~No initial charge~~

s-domain circuit:



Equation for I(s): $I(s)=\dfrac{32/s}{R+sL+\dfrac{1}{sC}}$ with R=0.96, L=0.8, and C=1.25, then

$$I(s)=\frac{32/s}{0.96+0.8s+\dfrac{1}{1.25s}}=\frac{40}{s^2+1.2\cancel{5}s+1}$$

$\sin\omega t \rightarrow \dfrac{\omega}{s^2+\omega^2}$

Now let's change to an entry form:

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Use complex formula

$$I(s)=\frac{40}{s^2+1.2s+1}=\frac{40}{(s+0.6)^2+0.8^2}=\frac{50\cdot(0.8)}{(s+0.6)^2+0.8^2}$$

i(t): $i(t)=50e^{-0.6t}\sin 0.8t$, t>0

$$50\frac{0.8}{(s+0.6)^2-(j0.8)^2}=50\left(\frac{A+jB}{s+0.6-j0.8}+\frac{A-jB}{s+0.6+j0.8}\right)$$

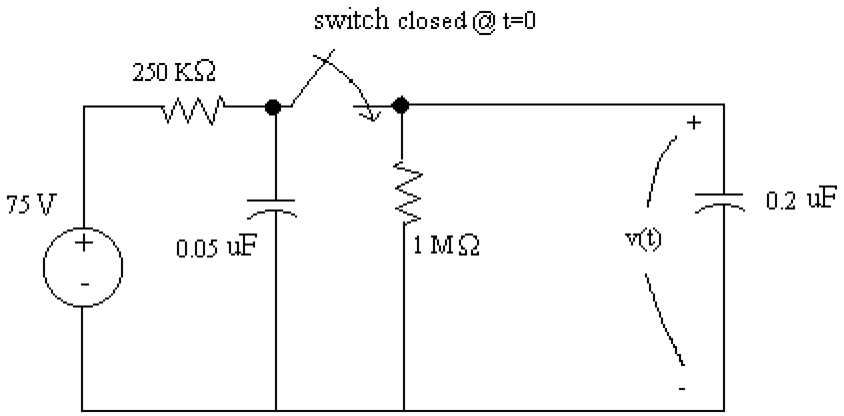
$\rightarrow A=0,\ B=-0.5 \rightarrow K=0.5\angle-90^\circ \qquad \theta=-90^\circ$

$\rightarrow 50\cdot 2|K|e^{-\alpha t}\cos(\beta t+\theta)$

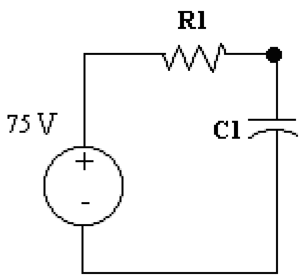
$=50\cdot 2\cdot(0.5)e^{-0.6t}\cos(0.8t-90^\circ)=$

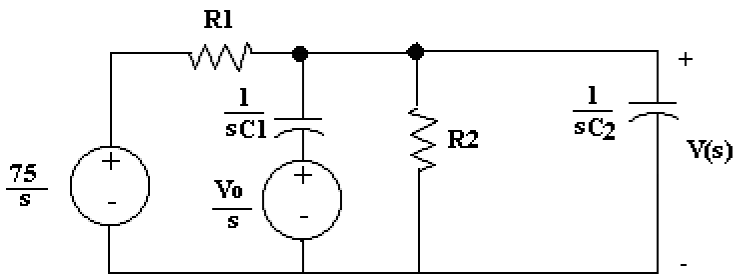
$=50e^{-0.6t}\sin(0.8t)$

2. The switch in the circuit has been opened for a long time. At t = 0 the switch closes. Find voltage v(t) by using s-domain analysis.



(a) t<0: Initial voltage across the capacitor C1 is, then, 75 [V]. V_o=75.





(b) t>0: s-domain circuit: (See above right)

Equation for V(s): node-voltage equation solves for V(s).

$$\frac{V(s)-75/s}{R_1}+\frac{V(s)-V_0/s}{1/sC_1}+\frac{V(s)}{R_2}+\frac{V(s)}{1/sC_2}=0$$

Arranging for V(s):

$$\frac{75}{sR_1}+C_1V_0=\frac{V(s)}{R_1}+sC_1V(s)+\frac{V(s)}{R_2}+sC_2V(s)$$

With, R1=250K, R2=1M, C1=0.05uF, and C2=0.2uF, equation above becomes,

× 20 $$\left(\frac{300}{s}+3.75\right)10^{-6}=10^{-6}\left(\frac{1}{0.25}+\frac{s}{20}+1+\frac{s}{5}\right)V(s)$$

4

Finally, $\left(\frac{6000}{s}+75\right)=(80+s+20+4s)V(s)$

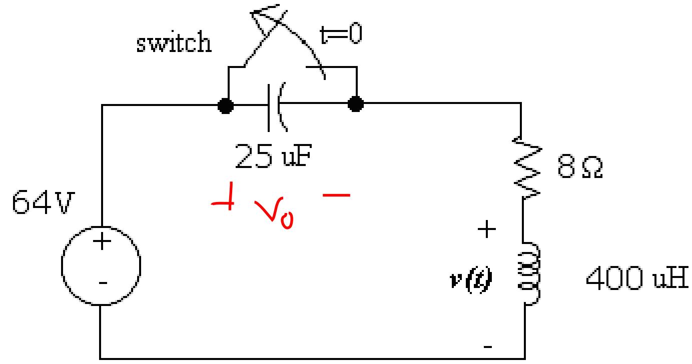
Then, $V(s)=\dfrac{\frac{6000}{s}+75}{5s+100}=\dfrac{75s+6000}{5s(s+20)}=\dfrac{15s+1200}{s(s+20)}=\dfrac{A}{s}+\dfrac{B}{s+20}$

By residue method: A=60 and B= - 45

Therefore, $V(s)=\dfrac{60}{s}-\dfrac{45}{s+20}$

v(t): $v(t)=60u(t)-45e^{-20t}$, t>0

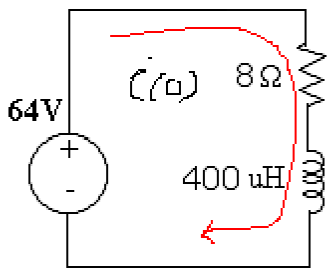
3. The switch in the circuit shown below has been closed for a long time. At t=0, the switch is opened. Find $v(t)$ by inverse Laplace transformation of $V(s)$.

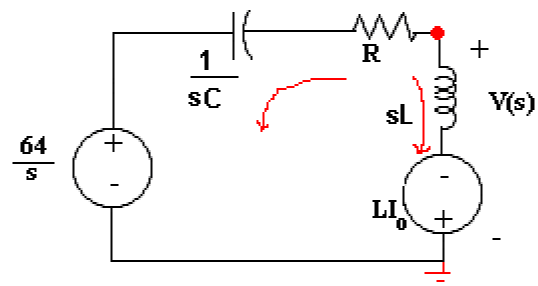


target $v(t)$ → $V(s)$

(a) t<0 $\quad I_o = 8, \; V_o = 0$

(b) t>0: s-domain circuit:





Equation for V(s): node voltage method:

$$\frac{V(s)+LI_o}{sL}+\frac{V(s)-64/s}{R+1/sC}=0 \;\Rightarrow V(s)\left[\frac{1}{sL}+\frac{1}{R+1/sC}\right]=\frac{64/s}{R+1/sC}+\frac{LI_o}{sL}$$

Therefore,

$$V(s)=sL\cdot\frac{64C+LI_oCs}{LCs^2+RCS+1}-LI_o=\frac{64LCS+L^2I_oCs^2-L^2I_oCs^2-LRCI_os-LI_o}{LCs^2+RCS+1}$$

which simplifies to

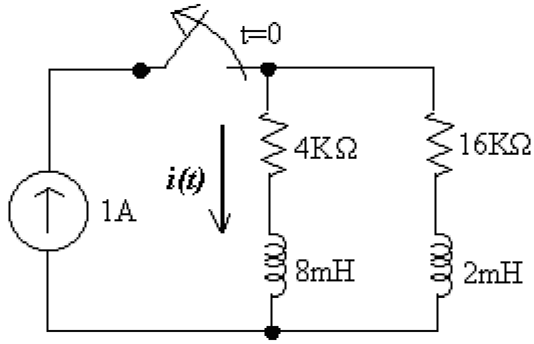
$$V(s)=\frac{64LCS-LRCI_os-LI_o}{LCs^2+RCS+1}=\frac{-\frac{I_o}{C}}{s^2+\frac{R}{L}s+\frac{1}{LC}}$$ (because I_oR=64)

With $\frac{R}{L}=\frac{8}{400\times10^{-6}}=2\times10^4$, $\frac{I_o}{C}=\frac{8}{25\times10^{-6}}=32\times10^4$, and $\frac{1}{LC}=10^8$

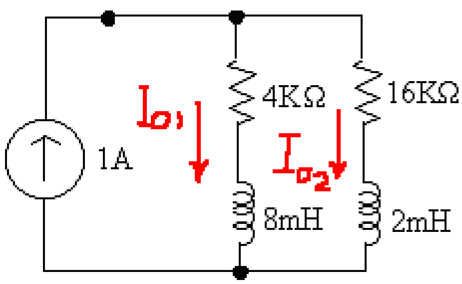
$$V(s)=\frac{-32\times10^4}{s^2+2\times10^4s+10^8}=\frac{-32\times10^4}{\left(s+10^4\right)^2}$$

v(t): $v(t)=-32\times10^4\cdot t\cdot e^{-10^4t}$, t>0

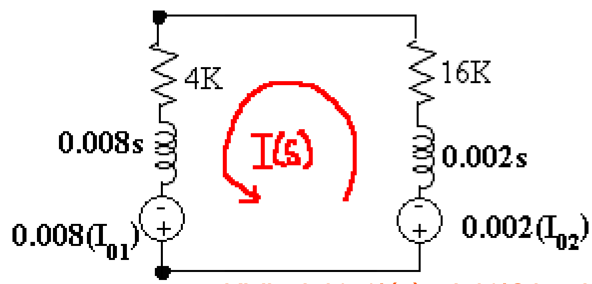
4. The switch below has been closed for a long time. At t = 0 the switch opens. Find i(t) by inverse Laplace transformation of I(s).



(a) t<0



(b) t>0



KVL: L1*s*I(s) - L1*I01 + L2*I02 + L2*s*I(s) + I(s)*20000=0

$$I_{o1} = 1 \cdot \frac{16}{4+16} = 0.8 \text{ and } I_{o2} = 1 \cdot \frac{4}{4+16} = 0.2$$

Equation for I(s): $I(s) = \dfrac{(0.008)(0.8)-(0.002)(0.2)}{(0.008+0.002)s+20000} = \dfrac{0.006}{0.01s+20000} = \dfrac{0.6}{s+2000000}$

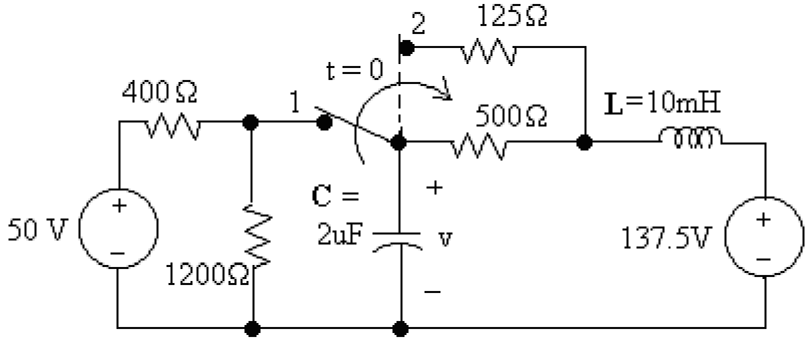
i(t): $i(t) = 0.6e^{-2\times10^{-6}t}$, t>0

L1*I01 - L2*I01

I(s) = -----------------------------

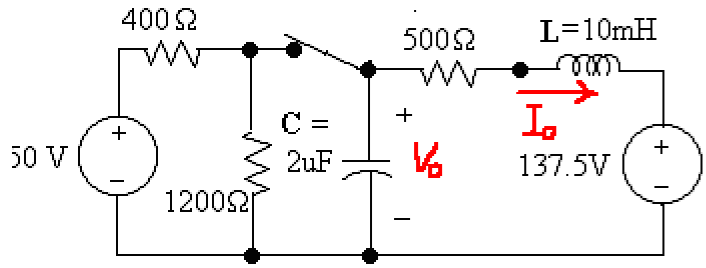
(L1+L2)*s + 20000

5. The switch in the circuit seen in the figure below has been in position 1 for a long time. At t = 0 it moves instantaneously from 1 to 2 position. Find v(t) by inverse transformation of s-domain voltage $V(s)$



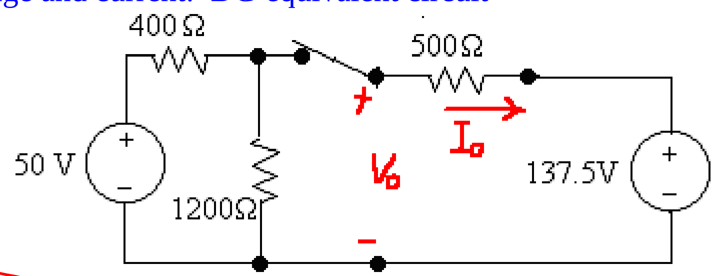
SOLUTION

(a) t<0



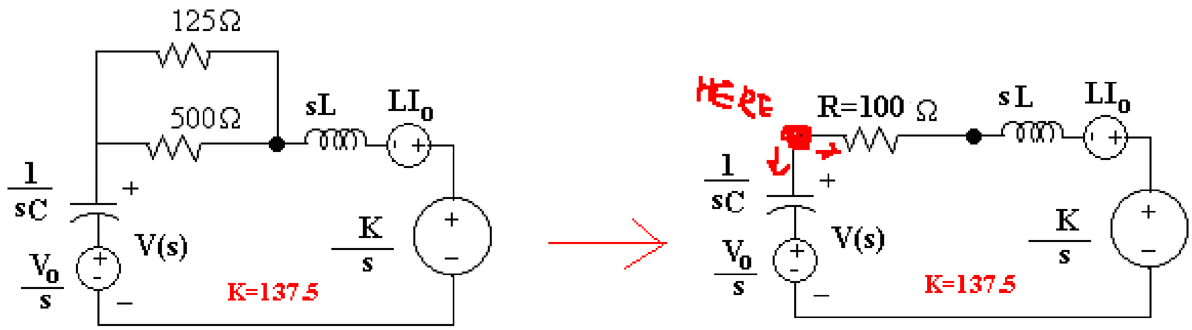
Finding initial voltage and current: DC-equivalent circuit

$$\frac{V0-50}{400} + \frac{V0}{1200} + \frac{V-137.5}{500} = 0$$



Therefore, $V_o = \cancel{50\frac{1200}{1600}} = 75$, $I_o = \frac{V_o - 137.5}{500} = \frac{75 - 137.5}{500} = -0.125$

(b) t>0 and s-domain circuit:



Equation for V(s):

Applying a node-voltage equation at the upper left corner:

$$sC[V(s) - \frac{V_o}{s}] + \frac{V(s) + LI_o - \frac{K}{s}}{R + sL} = 0$$

Then;

$$[sC+\frac{1}{R+sL}]V(s)=CV_o+\frac{-LI_o+\frac{K}{s}}{R+sL}=\frac{CV_o(R+sL)-LI_o+\frac{K}{s}}{(R+sL)}=\frac{sCV_oR+s^2CV_oL-sLI_o+K}{s(R+sL)}$$

Therefore,

$$V(s)=\frac{\frac{sCV_oR+s^2CV_oL-sLI_o+K}{s(R+sL)}}{sC+\frac{1}{R+sL}}=\frac{\frac{sCV_oR+s^2CV_oL-sLI_o+K}{s(R+sL)}}{\frac{1+sCR+s^2LC}{R+sL}}=\frac{sCV_oR+s^2CV_oL-sLI_o+K}{s(1+sCR+s^2LC)}$$

By arranging the denominator of the above function, we have:

$$V(s)=\frac{s^2V_o+(\frac{R}{L}V_o-\frac{I_o}{C})s+\frac{K}{LC}}{s(s^2+\frac{R}{L}s+\frac{1}{LC})}=\frac{75s^2+(75\times10^4+6.25\times10^4)s+6875\times10^6}{s(s^2+10^4s+50\times10^6)}$$

Partial Expansion:

$$V(s)=\frac{75s^2+812500s+6875\times10^6}{s(s^2+10^4s+50\times10^6)}=\frac{A}{s}+\frac{Bs+C}{s^2+10^4s+50\times10^6}$$

(s+5000)^2 +(5000)^2

(s+5000)^2-(j5000)^2=0

a^2 - b^2=(a+b)(a-b)

By residue method: $A=\cancel{\frac{6875\times10^6}{50\times10^6}=137.5}$

complex root

By algebraic method:

First we get this ~~my~~ b multiplication:

$$75s^2+812500s+6875\times10^6=A(s^2+10^4s+50\times10^6)+Bs^2+Cs$$

Then, from A+B=75, we get B=-62.5

Also, from C+1375000=812500, C=-562500

$A+B=75$

$10^4A+C=812500$

$50\cdot10^6A=6875\cdot10^6$

$A=137.5$

$B=-62.5$

Finally,

$$V(s)=\frac{137.5}{s}-\frac{62.5s+562500}{s^2+10^4s+50\times10^6}$$

$$=\frac{137.5}{s}-\frac{62.5(s+5000)+250000}{(s+5000)^2+25\times10^6}=\frac{137.5}{s}-\frac{62.5(s+5000)}{(s+5000)^2+5000^2}-\frac{50(5000)}{(s+5000)^2+5000^2}$$

v(t):

$$v(t)=137.5u(t)-62.5e^{-5000t}\cos5000t-50e^{-5000t}\sin5000t$$

OK.

$$=137.5u(t)-\sqrt{62.5^2+50^2}\,e^{-5000t}\cos(5000t-\arctan\frac{50}{62.5})$$

$$=137.5u(t)-80e^{-5000t}\cos(5000t-38^\circ)$$

for t>0

Using Complex Formula <Secondary>

$$\frac{+62.5S+562500}{S^2+10^4S+50\times10^6} = \frac{+62.5S+562500}{(S+5000)^2-(j5000)^2}$$

$$= \left(\frac{A+jB}{S+5000-j5000} + \frac{A-jB}{S+5000+j5000}\right) = \overbrace{\frac{A+jB}{(s+\alpha-j\beta)}}^{K} + \overbrace{\frac{A-jB}{(s+\alpha+j\beta)}}^{K^*}$$

By Algebraic

$$(A+jB)(S+5000+j5000)+(A-jB)(S+5000-j5000) = +62.5S+562500$$

$$2AS+10000A-10000B = 62.5S+562500$$

$$A = \frac{62.5}{2} = 31.25$$

$$\rightarrow B = \frac{1}{10000}\left(\frac{625000}{2} - 562500\right) = -25$$

$$\theta = -38^\circ$$

$$K = A+jB = 31.25-j25 \rightarrow K = 40\angle -38^\circ$$

$$K^* = 31.25+j25 \rightarrow K^* = 40\angle 38^\circ$$

$$|K| = 40$$

By the formula

$$\rightarrow 2|K|e^{-\alpha t}\cos(\beta t+\theta)$$

$$\rightarrow 2\cdot 40\cdot e^{-5000t}\cos(5000t-38^\circ)$$

$$= 80e^{-5000t}\cdot\cos(5000t-38^\circ)$$

4: Application of Laplace Transformation to Integrodifferential Equations

The Laplace transform is useful in solving linear integrodifferential equations by following:

1. Using the differential and integration properties, each term in the integrodifferetial equation is transformed. Initial conditions must be taken into account, though.
2. We solve the resulting algebraic equation in the *s* domain.
3. We then convert the solution back to the time domain by using the inverse Laplace transformation

Example 1:

Solve the differential equation using the Laplace transformation.

$$\frac{d^2v(t)}{dt^2} + 6\frac{dv(t)}{dt} + 8v(t) = 2u(t), \text{ subject to } v(0) = 1, v'(0) = -2.$$

Solution:

By taking Laplace transform of each term,

$$[s^2V(s) - sv(0) - v'(0)] + 6[sV(s) - v(0)] + 8V(s) = \frac{2}{s}$$

substituting $v(0) = 1, v'(0) = -2$,

$$s^2V(s) - s + 2 + 6sV(s) - 6 + 8V(s) = \frac{2}{s}$$

or, $$(s^2 + 6s + 8)V(s) = s + 4 + \frac{2}{s}$$

Hence, $V(s) = \dfrac{s^2 + 4s + 2}{s(s^2 + 6s + 8)} = \dfrac{s^2 + 4s + 2}{s(s+2)(s+4)} = \dfrac{A}{s} + \dfrac{B}{s+2} + \dfrac{C}{s+4}$

By residue or algebraic method, we have: A=0.25, B=0.5, and C=0.25

Therefore, $v(t) = 0.25[u(t) + 2e^{-2t} + e^{-4t}]$, t>0

Example 2:

Solve for y(t) in the following integrodifferntial equation.

$$\frac{dy(t)}{dt} + 5y(t) + 6\int_0^t y(x)dx = u(t), \text{ y(0)=2.}$$

Solution:

Taking Laplace transformation of each term,

$$[sY(s) - y(0)] + 5Y(s) + 6\frac{Y(s)}{s} = \frac{1}{s}$$

Substituting y(0)=2 and multiplying through by *s*,

$$Y(s)[s^2 + 5s + 6] = 1 + 2s,$$

or, $$Y(s) = \frac{1 + 2s}{s^2 + 5s + 6} = \frac{-3}{s+2} + \frac{5}{s+3}$$

Thus, $y(t) = -3e^{-2t} + 5e^{-3t}$, t>0

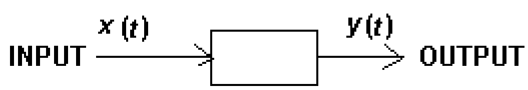
5: System Level *s*-domain Analysis

A. System Consideration

So far we analyzed circuits using s-domain analysis and now it's time to broaden our analysis to the systems level. In the system level analysis, mathematical Input-Output Relationship is more important than the circuit details. This system analysis will allow you to deal with many of the basic concepts of control and communication systems.

B. Transfer Function

In our circuit analysis, we found load voltage (or current) of a circuit excited by sources. In the system level analysis, the details of the circuit could be replaced by just a box (or "black box"). Then, the excitation is the INPUT to the box, and the response, the OUTPUT from the box. A means to describe the "black box" using x(t) and y(t), without knowing the details inside the box is a function called, "Transfer Function," which literally transfers x(t) to y(t). In the diagram below, the transfer function could be a function, *g*(t). Then the output y(t) is derived by the convolution (not the scope of this course) of the input and the transfer function: $y(t) = x(t) * g(t)$.



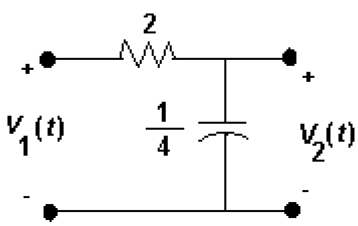
In **s**-domain, with $X(s) = L\{x(t)\}$ and $Y(s) = L\{y(t)\}$, the diagram above changes to:

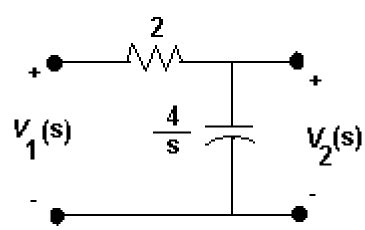
Then, in s-domain, the output Y(s) can be derived by the simple multiplication of the input and the transfer function: $Y(s) = X(s)G(s)$. This is one beauty of the *s*-domain analysis. However, s-domain and time-domain equations are actually equivalent.

<u>Transfer Function Example</u>

(a) Determine Transfer function of the circuit (below left).

(b) Using the transfer function, determine the response due to an input source $v_1(t) = 5\sin 2t$.





SOLUTION

(a) Convert the circuit to *s*-domain (above right)

Transfer function: $G(s)=\dfrac{V_2(s)}{V_1(s)}=\dfrac{V_1(s)\left\{\dfrac{4/s}{2+4/s}\right\}}{V_1(s)}=\dfrac{4/s}{2+4/s}=\dfrac{2}{s+2}$

The time domain transfer function is the inverse transform of G(s): $g(t)=2.e^{-2t}$, t>0.

(b) The input of $v_1(t)=5\sin 2t$ in s-domain is: $V_1(s)=\dfrac{10}{s^2+4}$

Therefore, output is: $V_2(s)=V_1(s)G(s)=\dfrac{2}{s+2}\cdot\dfrac{10}{s^2+4}$

Let's inverse transform of the output:

$$V_2(s)=\frac{2}{s+2}\cdot\frac{10}{s^2+4}=\frac{A}{s+2}+\frac{Bs+C}{s^2+4}$$

By residue method: A=5/2

By algebraic method: C=5, and B= - 5/2

Therefore,

$$V_2(s)=\frac{2}{s+2}\cdot\frac{10}{s^2+4}=\frac{5/2}{s+2}+\frac{-5/2s+5}{s^2+4}=\frac{5}{2}\left[\frac{1}{s+2}-\frac{s}{s^2+4}+\frac{2}{s^2+4}\right]$$

Finally, the inverse transform gives:

$$v_2(t)=\frac{5}{2}\left[e^{-2t}-\cos t2t+\sin 2t\right]=\frac{5}{2}e^{-2t}+\frac{5}{\sqrt{2}}\cos(2t+45)\text{, t>0}$$

(c) Observation of the output

The output has two terms.The first term is an exponential decaying **transient** one, and the second one is **steady-state**. The transient term is due to the circuit (or "**natural behavior of the circuit**") and the steady-state term is **due to the input** source.

C. Poles and Zeros

The transfer function G(s) is usually expressed by the polynomials of numerator and denominator: $G(s)=\dfrac{N(s)}{D(s)}$. Then **Poles** are defined as the roots of D(s) and the **zeros** are the roots of N(s). In other words, **poles** are the values of *s* that will cause the transfer function to be infinity (∞), while **zeros** cause it to be zero (0).

The variable *s* is a complex variable, so it can be expressed by

$$s=\sigma+j\omega$$

where, σ is the damping constant, and ω, angular frequency.

From the above example of $V_2(s)=\dfrac{2}{s+2}\cdot\dfrac{10}{s^2+4}$, there are 3 poles at $s=-2,-j2,+j2$.

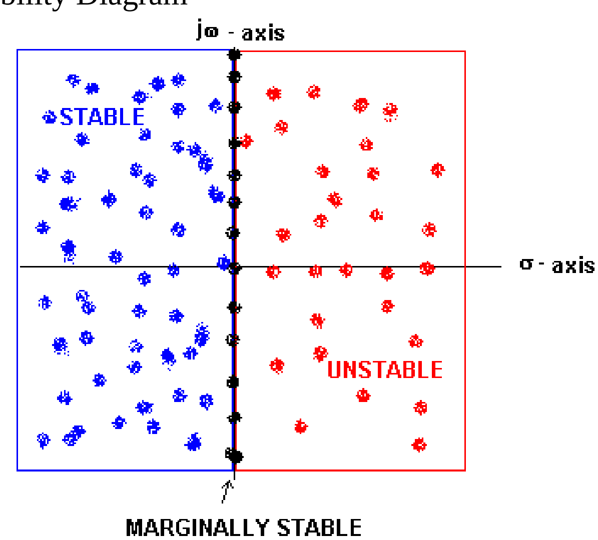
There is no zero. (or we can say there are 3 (the same number as the poles) zeros at "infinity")

Pole Location

The location of Poles (marked by "x") in complex s-plane indicate the system behavior (or "stability") while that of zeros (marked by "o") does not ordinarily affect. So let's discuss about the system stability with pole locations in the s-plane.

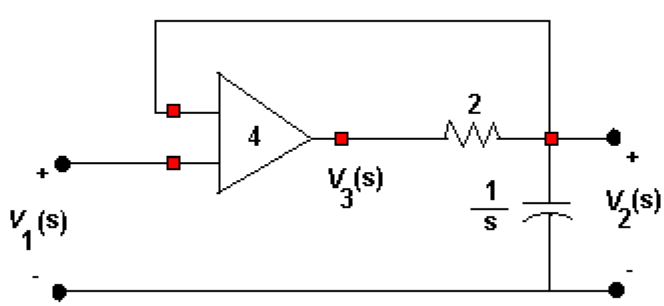
Pole Location	**F(s)**	**f(t)**	**System Stability**
On negative real axis	$\dfrac{A}{s+a}$	Ae^{-at}	Stable
On negative real plane	$\dfrac{A}{[s+a+jw][s+a-jw]}=\dfrac{A}{(s+a)^2+w^2}$	$Ae^{-at}\sin wt$	Stable
On positive real axis	$\dfrac{A}{s-a}$	Ae^{at}	Unstable
On positive real plane	$\dfrac{A}{(s-a)^2+w^2}$	$Ae^{at}\sin wt$	Unstable
At the origin	$\dfrac{A}{s}$	$Au(t)$	Marginally Stable
On jw-axis	$\dfrac{A}{s^2+w^2}$	$B\sin wt$	Marginally Stable

Pole Location/System Stability Diagram



<u>Stability Check Example</u>

Show the following active amplifier circuit (in *s*-domain) is unstable by the poles locations. The OP Amp behaves as $V_3(s)=4[V_1(s)+V_2(s)]$.



SOLUTION

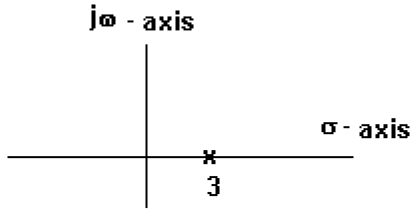
Since current does not flow to the Op Amp, the node voltage equation at the output terminal is:

$$\frac{V_2(s)-V_3(s)}{1}+\frac{V_2(s)}{1/s}=0$$

Substituting the relationship of $V_3(s)=4[V_1(s)+V_2(s)]$ yields, $V_2(s)(s-3)=4V_1(s)$

Therefore, the transfer function is: $G(s)=\dfrac{V_2(s)}{V_1(s)}=\dfrac{4}{s-3}$

The pole location is on the positive real axis and the time domain function $g(t)=4e^{3t}$ is exponentially increasing. System is unstable.



D. Steady-State Transfer Function and Frequency Response

As we discussed before, the variable *s* is a complex number, with damping (transient) component and the steady-sate component. If we are interested only in the state-steady condition of a sinusoidal system, we can change the variable *s* to be a complex number without real part. In other words, in steady-state, $s = j\omega$. In the system behavior analysis, if poles are on the imaginary axis, system is marginally stable (meaning that steady state). Therefore, the transfer function of steady-state can be equated as: $G(s)=G(jw)$. By doing this, we suddenly found us in the frequency-domain, and the frequency response world.

The steady-state transfer function is a complex number and a function of the angular frequency, so it can be expressed by the amplitude and the phase angle:

$$G(jw)=A(w)e^{j\theta(w)}=A(w)\angle\theta(w)$$

$A(w)$: Amplitude response, and $\theta(w)$: Phase response

This analysis opens the "frequency response." Consider a transfer function $G(jw)$ and an input signal $x(t)=V_{\max}\cos(200t+\phi)$. Note that the input signal is a single-frequency sinusoid.

Then, the output is: $Y(jw)=G(jw)X(jw)=A(w)\angle\theta(w)\cdot V_{\max}\angle\phi=A(w)V_{\max}\angle[\theta(w)+\phi]$

Therefore, the output amplitude response is: $Y=A(w)V_{\max}$, and

the output phase response is: $\gamma=\theta(w)+\phi$

Or, we can express the transfer function response in terms of the input and output:

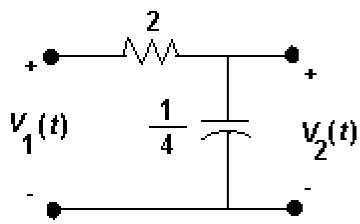
$$A(w)=\frac{Y}{V_{\max}} \text{ and } \theta(w)=\gamma-\phi$$

The above expression of amplitude is called "absolute amplitude" and there is much convenient expression of amplitude, "**relative amplitude**," which is defined as:

$A_{dB}(w) = 20\log_{10}\frac{A(w)}{A(0)}$,where $A(0)$ is the reference amplitude at a particular frequency (0 Hz)

<u>Frequency Response Measurement Example</u>

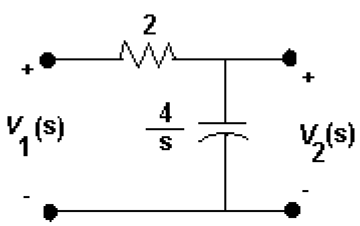
For the circuit below,



(a) Determine the steady-sate transfer function of the circuit for any arbitrary *w*.
(b) Determine the steady-state response due to the input signal $v_1(t) = 5\sin 2t$ ***by means of phasor concept.***
(c) Determine the steady-state response if the angular frequency of the input signal is changed to *w*=10 rad/s.
(d) Using the amplitude response at dc (0 Hz) as a reference, determine the relative dB loss (or gain) at *w*=2 and *w*=10.

SOLUTION

The s-domain circuit:



(a) From earlier example, $G(s) = \frac{2}{s+2}$, therefore, $G(jw) = \frac{2}{2+jw}$

And, $A(w) = \frac{2}{\sqrt{4+w^2}}$ and $\theta(w) = -\tan^{-1}\frac{w}{2}$

(b) Since $5\sin 2t = 5\cos(2t-90) = 5\angle -90 = -j5$

Therefore, by voltage division,

$$V_2(j2) = V_1(j2)\frac{4/j2}{2+4/j2} = (-j5)\frac{4/j2}{2+4/j2} = \frac{-j20}{4+j4} = \frac{20\angle -90}{4\sqrt{2}\angle 45} = \frac{5}{\sqrt{2}}\angle -135$$

Therefore, $v_2(t) = 3.536\cos(2t-135)$ [for w=2 case]

(c) $A(10) = \frac{2}{\sqrt{4+10^2}} = 0.196$, and $\theta(10) = -\tan^{-1}\frac{10}{2} = -78.69^\circ$

Therefore, $V_2(j10) = G(j10)V_1(j10) = [0.196\angle -78.69][5\angle -90]$

NOTE: Since $v_i(t)$ is a single frequency signal, so $V_1(jw) = V_1(j2) = -j5$ for any w.

Finally, $v_2(t) = 0.981(\cos t10t - 168.69)$

STOP: Observation: Do you see the magnitude change between the results of (b) and (c) when the frequency changes from w=2 to w=10?

(d) At w=10, $A(w) = \dfrac{2}{2 + j0} = 1 = A(0)$

@w=2: relative dB gain (loss) is:

$$A_{dB}(2) = 20\log_{10}\frac{A(2)}{A(0)} = 20\log_{10}\left(\frac{2}{\sqrt{4+4}}\right) = 20\log_{10}\frac{1}{\sqrt{2}} = -10\log_{10}2 = -3.01\,[\text{dB}]$$

@w=10: relative dB gain (loss) is:

$$A_{dB}(10) = 20\log_{10}\frac{A(10)}{A(0)} = 20\log_{10}\left(\frac{2}{\sqrt{4+100}}\right) = 20\log_{10}0.196 = -14.15\,[\text{dB}]$$